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Aharonov-Bohm Effect vs. Dirac Monopole: A-B ⇔ D

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Abstract In the context of fiber bundle theory, we show that the existence of the Aharonov-Bohm connection implies the existence and uniqueness of the Dirac connection.

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1 Introduction

As is well known, the Aharonov-Bohm $(A-B)$ effect [1,2], crucial for exhibiting the non local character of quantum mechanics and for the development of the gauge theories of fundamental interactions, and the Dirac monopole (D) proposal [3,4], which implies the quantization of electric charge once a single monopole exists in Nature, are intimately related.

A hypothetical static magnetic charge g at the origin of $\mathbb{R}^3$ has a magnetic field

$$\boldsymbol{B} = g\frac{\boldsymbol{r}}{r^3} \tag{1}$$

which, as is known, cannot be derived from a unique vector potential $\boldsymbol{A}$ in the whole space through the usual formula

$$\boldsymbol{B} = rot(\boldsymbol{A}). \tag{2}$$

This leads to a zero magnetic flux around g, contrary to its value $4\pi g$. Instead, in polar coordinates [5],

$$\boldsymbol{A}_{\pm} = \pm\frac{g((1\mp cos\theta)}{rsin\theta}\hat{\varphi} \tag{3}$$

give $\boldsymbol{B}$ at all points of space except at the semi-axis $\theta = \pi$ $(z \leq 0)$ for $\boldsymbol{A}_+$, and the semi-axis $\theta = 0$ $(z \geq 0)$ for $\boldsymbol{A}_-$. Each excluded region is known as a string of singularities (Dirac string). The difference between the two potentials is a gradient:

$$\boldsymbol{A}_+ - \boldsymbol{A}_- = \nabla\Lambda,\ \Lambda = 2g\varphi \tag{4}$$

namely, a gauge transformation. By the gauge invariance of electrodynamics, a quantum particle with electric charge e moving in the above magnetic field should not "see" the strings and so its wave function ψ should be multiplied by the phase factor $exp(\frac{ie}{\hbar c}\Lambda)$ i.e.

$$\psi \to \psi' = exp(\frac{i2ge\varphi}{\hbar c})\psi. \tag{5}$$

The single valued character of the wave function requires that $\psi'(\varphi+2\pi) = \psi'(\varphi)$ i.e.

$$\frac{2ge}{\hbar c} = n,\ n \in \mathbb{Z} \tag{6}$$

which is the Dirac quantization condition (DQC) for electric charge: $e = e_n = n \times \frac{1}{2g}$ in natural units $\hbar = c = 1$.

On the other hand, the semi classical approximation to the $A-B$ effect [6], which consists in considering the contribution of only two classical paths to the path integral describing the passage of an electric charge e from its source through two slits and later surrounding an impenetrable solenoid of radius R (in the z direction), with enclosed magnetic flux $\Phi_{A-B}=\pi R^2|\boldsymbol{B}|$ with constant field

$$\boldsymbol{B}=rot(\boldsymbol{A})=\{\begin{matrix}|\boldsymbol{B}|\hat{z}, \ r\leq R\\ 0, \ r>R\end{matrix} \tag{7}$$

and potential

$$\boldsymbol{A}=\{\begin{matrix}\frac{|\boldsymbol{B}|r}{2}\hat{\varphi}, \ r\leq R\\ \frac{\Phi_{A-B}}{2\pi r}\hat{\varphi}, \ r>R\end{matrix}, \tag{8}$$

leads to an interference pattern on a last stage screen, shifted with respect to that without the magnetic field in the phase factor

$$exp(2\pi i\frac{\Phi_{A-B}}{\Phi_0}), \tag{9}$$

where

$$\Phi_0=\frac{2\pi\hbar c}{|e|} \tag{10}$$

($=\frac{2\pi}{|e|}$ in natural units) is the quantum of magnetic flux associated with the charge e. The $A-B$ effect "disappears" when $\Phi_{A-B}=n\Phi_0=2\pi n\frac{\hbar c}{|e|}$, that is, when the condition

$$|e|\Phi_{A-B}=2\pi n\hbar c \tag{11}$$

holds. This is nothing but the DQC when Φ_{A-B} equals the flux $4\pi g$ of a magnetic charge g.

So, the quantum "invisibility" of the Dirac string amounts to the "disappearance" of the $A-B$ effect. This is the physical relation between the two phenomena.

Given the topological non triviality of some characteristics of the $A-B$ effect -for an idealized infinite long and infinitely thin solenoid the relevant space is the punctured plane $\mathbb{C}^*$-, and of the D magnetic charge -non existence of a global potential-, the appropriate description of the phenomena is in the context of the theory of fiber bundles and connections [7]. In particular the $U(1)$-bundles: i) $\xi_{A-B}:\mathbb{C}^*\times U(1)\to\mathbb{C}^*$ in the $A-B$ case, and ii) the Hopf bundle $\xi_D:S^3\to S^2$ for the magnetic charge $g=1/2$. The underlying relation between these bundles and connections has not been, according to the author's knowledge, yet discussed in the literature. In ref. [8] it was shown that: i) ξ_{A-B} is isomorphic to the pull-back of ξ_D induced by the inclusion of the corresponding base spaces, $\mathbb{C}^*$ and $S^2\cong\mathbb{C}\cup\{\infty\}$, and ii) the D connection on ξ_D evaluated on the equator $\theta=\pi/2$ gives the $A-B$ connection on ξ_{A-B}. So, in this sense, the D connection implies the existence of the $A-B$ connection. The aim of the present work is to prove the inverse implication.

Details of the geometric description is presented in Section **2**. In Sections **3** and **4**, through the construction of a sort of "truncation" $\hat{\xi}_D$ of ξ_D, and the processes of pulling-back and pushing-forward respectively the $A-B$ potentials on $\mathbb{C}^*$ and the $A-B$ connection A_{A-B} on $\mathbb{C}^*\times U(1)$, we obtain the restriction $\omega_D|$ on $\hat{\xi}_D$ of the D connection ω_D on ξ_D, evaluated at the equator $\theta=\pi/2$ i.e. $\omega_D|(\theta=\pi/2)$. Finally, in Section **5**, using symmetry arguments, we recover the whole ω_D from A_{A-B} through the *unique* θ-dependent extension of $\omega_D|(\theta=\pi/2)$.

2 Geometric Description

The Aharonov-Bohm effect ($A-B$) [1,2] can be described in the trivial $U(1)$-bundle [9]

$$\xi_{A-B}:U(1)\hookrightarrow\mathbb{C}^*\times U(1)\xrightarrow{pr_1}\mathbb{C}^*, \qquad pr_1(z,\mu)=z \tag{12}$$

($\mathbb{C}^*=\mathbb{C}\setminus\{0\}$, $\mu=e^{i\varphi}$, $\varphi\in[0,2\pi)$), while the hypotetical $g=\frac{1}{2}$ magnetic charge or Dirac monopole (D) [3,4] can be described in the non trivial $U(1)$-bundle [10]

$$\xi_D:U(1)\hookrightarrow S^3\xrightarrow{\pi_H}S^2 \tag{13}$$

(Hopf bundle, [11]), where π_H is the Hopf map

$$\pi_H(z_1, z_2) = \{ \begin{matrix} z_1/z_2, & z_2 \neq 0 \\ \infty, & z_2 = 0 \end{matrix} \tag{14}$$

with

$$S^3 = \{(z_1, z_2)| \ \ |z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2 \tag{15}$$

and

$$\Phi : S^2 \subset \mathbb{R}^3 \xrightarrow{\cong} \mathbb{C} \cup \{\infty\}, \ \ \Phi(x_1, x_2, x_3) = \{ \begin{matrix} \frac{x_1+ix_2}{1+x_3}, & (x_1, x_2, x_3) \neq (0, 0, -1) \\ \infty, & (x_1, x_2, x_3) = (0, 0, -1) \end{matrix} . \tag{16}$$

In terms of the Euler angles in $\mathbb{R}^3$, $\chi, \varphi \in [0, 2\pi)$, $\theta \in [0, \pi]$, the $u(1) = Lie(U(1))$-valued D connection on S^3 is given by [12]

$$\omega_D = \frac{i}{2}(d\chi + cos\theta \ d\varphi) \tag{17}$$

with D potentials on S^2

$$A_{D\pm} = \mp \frac{i}{2}(1 - cos\theta) \ d\varphi, \tag{18}$$

and curvature $F_D = d\omega_D = \frac{i}{2} sin\theta \ d\theta \wedge d\varphi$: $(-i)\times$ the magnetic field of the monopole, while the $A - B$ potentials on $\mathbb{C}^*$ (and global connection A_{A-B} on $\mathbb{C}^* \times U(1)$ since ξ_{A-B} is trivial) are given by [9]

$$A_{A-B\pm} = \mp \frac{i}{2} d\varphi = \mp \frac{i}{2} \frac{X_1 dX_2 - X_2 dX_1}{X_1^2 + X_2^2} \tag{19}$$

with $z = X_1 + iX_2 \in \mathbb{C}^*$ and X_1, X_2 the Cartesian coordinates on $\mathbb{R}^{2^*} \cong \mathbb{C}^*$; clearly, $A_{A-B\pm}$ are closed (A_{A-B} is flat in its domain of definition, $z \neq 0$) but not exact 1-forms.

From (18) and (19),

$$A_{D\pm}|_{\theta=\pi/2} = A_{A-B\pm} \tag{20}$$

which, in the context of bundle theory, tells us that the existence of the Dirac monopole implies the existence of the $A - B$ effect ("$D \Rightarrow A - B$"). The same conclusion has been arrived at in ref. [8], where the close relation between both phenomena was exhibited by showing that the $A-B$ bundle is equivalent (isomorphic) to the pull-back of the D bundle by the inclusion $\iota : \mathbb{C}^* \to \mathbb{C} \cup \{\infty\}$, $\iota(z) = z$, between the corresponding base spaces:

$$\xi_{A-B} \cong \iota^*(\xi_D). \tag{21}$$

This fact immediately raises the question for the inverse implication, namely, if the existence of the $A - B$ effect implies, at least in the present mathematical sense, the existence of the Dirac monopole [13]. These monopoles, though yet not found in Nature, are predicted by grand unified [14] and string [15] theories. The purpose of the present note is to answer affirmatively the above question.

3 Pull-Back of the $A - B$ Potentials

If $N = (0, 0, 1)$ and $S = (0, 0, -1)$ are the north and south poles of S^2, then

$$\pi_H^{-1}(\{N, S\}) = \{(z_1, 0), \ |z_1| = 1\} \cup \{(0, z_2), \ |z_2| = 1\} \cong S^1 \times S^1 = T^2, \tag{22}$$

the 2-torus. If we "truncate" the ξ_D bundle by defining the $U(1)$-bundle

$$\hat{\xi_D} : U(1) \hookrightarrow S^3 \setminus T^2 \xrightarrow{\pi_H|} S^2 \setminus \{N, S\} \cong \mathbb{C}^*, \tag{23}$$

the inclusion $\iota : \mathbb{C}^* \to S^2$ becomes the identity $Id_{\mathbb{C}^*}$ and we have the bundle map given by Diagram 1:

$$\begin{matrix} (\mathbb{C}^* \times U(1)) \times U(1) & \overset{\bar{\iota} \times Id_{U(1)}}{\longrightarrow} & (S^3 \setminus T^2) \times U(1) \\ \psi_0 \downarrow & & \downarrow \psi_D| \\ \mathbb{C}^* \times U(1) & \overset{\bar{\iota}}{\longrightarrow} & S^3 \setminus T^2 \\ pr_1 \downarrow & & \downarrow \pi_H| \\ \mathbb{C}^* & \overset{Id_{\mathbb{C}^*}}{\longrightarrow} & \mathbb{C}^* \end{matrix}$$

Diagram 1

where ψ_0 and $\psi_D|$ are the right actions of $U(1)$ on the corresponding total spaces,

$$\bar{\iota}(z,\mu) = \frac{(z,1)\mu}{||(z,1)||}, \tag{24}$$

and $|$ denotes the corresponding restrictions. It is clear that the "transitions" from ω_D and $A_{D\pm}$ to the restrictions $\omega_D|$ and $A_{D\pm}|$ respectively on $S^3 \setminus T^2$ and $\mathbb{C}^*$ are continuous, since they amount to the restriction of the domain of θ from $[0,\pi]$ to $(0,\pi)$.

Defining Hopf coordinates [16] $\{\eta,\xi_1,\xi_2\}$ on S^3:

$$(z_1,z_2) = (e^{i\xi_1} sin\ \eta, e^{i\xi_2} cos\ \eta),\ \eta\in[0,\pi/2],\ \xi_1,\xi_2\in[0,2\pi] \tag{25}$$

we obtain

$$\pi_H|(\eta,\xi_1,\xi_2) = e^{i(\xi_1-\xi_2)} tg\ \eta, \tag{26}$$

with $\eta \in (0,\pi/2)$, which allows us to construct the pull-back $\beta \in \Omega^1(S^3\setminus T^2;u(1))$ of $A_{A-B\pm} \in \Omega^1(\mathbb{C}^*;u(1))$ by $\pi_H|^*$:

$$\begin{pmatrix}\beta_\eta\\ \beta_{\xi_1}\\ \beta_{\xi_2}\end{pmatrix} = \pm\begin{pmatrix}\frac{\partial}{\partial\eta}(X_1\circ\pi_H|) & \frac{\partial}{\partial\eta}(X_2\circ\pi_H|)\\ \frac{\partial}{\partial\xi_1}(X_1\circ\pi_H|) & \frac{\partial}{\partial\xi_1}(X_2\circ\pi_H|)\\ \frac{\partial}{\partial\xi_2}(X_1\circ\pi_H|) & \frac{\partial}{\partial\xi_2}(X_2\circ\pi_H|\end{pmatrix}\begin{pmatrix}A_{A-B1\pm}\\ A_{A-B2\pm}\end{pmatrix}$$

$$= \pm\begin{pmatrix}\frac{cos(\xi_1-\xi_2)}{cos^2\eta} & \frac{sin(\xi_1-\xi_2)}{cos^2\eta}\\ -sin(\xi_1-\xi_2)tg\ \eta & cos(\xi_1-\xi_2)tg\ \eta\\ sin(\xi_1-\xi_2)tg\ \eta & -cos(\xi_1-\xi_2)tg\ \eta\end{pmatrix}\begin{pmatrix}A_{A-B1\pm}\\ A_{A-B2\pm}\end{pmatrix} = \begin{pmatrix}0\\ i/2\\ -i/2\end{pmatrix} \tag{27}$$

i.e.

$$\beta = \frac{i}{2}(d\xi_1 - d\xi_2). \tag{28}$$

From the relation between Hopf coordinates and Euler angles,

$$(e^{i\xi_1} sin\ \eta, e^{i\xi_2} cos\ \eta) = (e^{\frac{i}{2}(\varphi+\chi)} cos(\theta/2), e^{\frac{i}{2}(\varphi-\chi)} sin(\theta/2)) \tag{29}$$

one obtains

$$\beta = \frac{i}{2} d\chi \tag{30}$$

i.e.

$$\pi_H|^*(A_{A-B\pm}) = \omega_D|(\theta=\pi/2). \tag{31}$$

4 Push-Forward of the $A-B$ Connection

The same relation between A_{A-B} and $\omega_D|(\theta=\pi/2)$ can be arrived at through the more direct path of pushing forward horizontal spaces of A_{A-B} in ξ_{A-B} into horizontal spaces of $\omega_D|(\theta=\pi/2)$ in $\hat{\xi_D}$. Since $Id_{\mathbb{C}^*}$ is a diffeomorphism and $Id_{U(1)}$ is a group homomorphism (isomorphism), we are in the conditions of Proposition 6.1. in ref. [17]: given A_{A-B} in ξ_{A-B} there *exist* and is *unique* a connection ω in $\hat{\xi_D}$ such that the horizontal subspaces of A_{A-B} in $\mathbb{C}^*\times U(1)$ are mapped into the horizontal subspaces of ω in $\hat{\xi_D}$ by $d\bar{\iota} \equiv \bar{\iota}_*$. Here, we shall explicitly prove this fact and find that $\omega = \omega_D|(\theta=\pi/2)$.

At any point $(z,e^{i\varphi})$ of $\mathbb{C}^*\times U(1)$, the horizontal space of A_{A-B} is the kernel of (19). So, $(X_1dX_2 - X_2dX_1)(V_1\frac{\partial}{\partial X_1} + V_2\frac{\partial}{\partial X_2}) = X_1V_2 - X_2V_1 = 0$ implies

$$V_2 = \frac{X_2}{X_1}V_1\ for\ \ X_1\neq 0,\ and\ V_1 = 0\ for\ X_1 = 0. \tag{32}$$

Since

$$T_{(z,e^{i\varphi})}(\mathbb{C}^*\times U(1)) = T_z\mathbb{C}^* \oplus T_{e^{i\varphi}}U(1) = \mathbb{C}\oplus\{te^{i(\varphi+\pi/2)}\}_{t\in\mathbb{R}}, \tag{33}$$

the horizontal vectors at $(z, e^{i\varphi})$ are given by

$$V \equiv (V_1, V_2, V_\varphi) = \{ \begin{matrix} (V_1, \frac{X_2}{X_1} V_1, ite^{i\varphi}), \ X_1 \neq 0 \\ (0, V_2, ite^{i\varphi}), \ X_1 = 0 \end{matrix} . \tag{34}$$

On the other hand, from the definition of $\bar{\iota}$ in eq. (24) and the definition of the Hopf coordinates on S^3, eq. (25), one obtains

$$\bar{\iota}(z, e^{i\varphi}) = \bar{\iota}(X_1 + iX_2, e^{i\varphi}) \cong \bar{\iota}(X_1, X_2, e^{i\varphi}) = (\eta(X_1, X_2, \varphi), \xi_1(X_1, X_2, \varphi), \xi_2(X_1, X_2, \varphi))$$

$$= (tg^{-1}(\sqrt{X_1^2 + X_2^2}), tg^{-1}(\frac{X_2 cos\ \varphi + X_1 sin\ \varphi}{X_1 cos\ \varphi - X_2 sin\ \varphi}), \varphi), \tag{35}$$

leading to $W = \bar{\iota}_*(V)$ with components

$$\begin{pmatrix} W_\eta \\ W_{\xi_1} \\ W_{\xi_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial \eta}{\partial X_1} & \frac{\partial \eta}{\partial X_2} & \frac{\partial \eta}{\partial \varphi} \\ \frac{\partial \xi_1}{\partial X_1} & \frac{\partial \xi_1}{\partial X_2} & \frac{\partial \xi_1}{\partial \varphi} \\ \frac{\partial \xi_2}{\partial X_1} & \frac{\partial \xi_2}{\partial X_2} & \frac{\partial \xi_2}{\partial \varphi} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_\varphi \end{pmatrix} = \begin{pmatrix} \frac{X_1}{(tg\ \eta)(1+tg^2\eta)} & \frac{X_2}{(tg\ \eta)(1+tg^2\eta)} & 0 \\ -\frac{X_2}{tg^2\eta} & \frac{X_1}{tg^2\eta} & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_\varphi \end{pmatrix} . \tag{36}$$

The relation between Hopf and Euler coordinates:

$$tg\ \eta = cotg(\frac{\theta}{2}), \ \xi_1 = \frac{\varphi + \chi}{2}, \ \xi_2 = \frac{\varphi - \chi}{2}, \tag{37}$$

allows to write

$$\omega_D(\eta, \xi_1, \xi_2) = \frac{i}{1 + tg^2\eta}(tg^2\eta\ d\xi_1 - d\xi_2). \tag{38}$$

In particular,

$$\omega_D(\pi/4, \xi_1, \xi_2) = \omega_D(\theta = \pi/2). \tag{38a}$$

The horizontal space at any point $(\eta, \xi_1, \xi_2) \in S^3 \setminus T^2$, being the kernel of $\omega_D|$, turns out to be

$$H_{(\eta,\xi_1,\xi_2)} = \{(v_\eta, v_1, (tg^2\eta)v_1), \ v_\eta, v_1 \in \mathbb{R}, \ \eta \in (0, \pi/2)\}. \tag{39}$$

In particular,

$$H_{(\pi/4,\xi_1,\xi_2)} = \{(v_{\pi/4}, v_1, v_1), \ v_{\pi/4}, v_1 \in \mathbb{R}\}. \tag{39a}$$

For $X_1 \neq 0$, (36) leads to

$$\begin{pmatrix} W_\eta \\ W_{\xi_1} \\ W_{\xi_2} \end{pmatrix} = \begin{pmatrix} \frac{V_1 tg\ \eta}{X_1(1+tg^2\eta)} \\ ite^{i\varphi} \\ ite^{i\varphi} \end{pmatrix} \tag{40}$$

while for $X_1 = 0$, (36) leads to

$$\begin{pmatrix} W_\eta \\ W_{\xi_1} \\ W_{\xi_2} \end{pmatrix} = \begin{pmatrix} \frac{X_2 V_2}{(tg\ \eta)(1+tg^2\eta)} \\ ite^{i\varphi} \\ ite^{i\varphi} \end{pmatrix}, \tag{41}$$

which belong to $H_{(\pi/4,\xi_1,\xi_2)}$. So, horizontal spaces of A_{A-B} are mapped into horizontal spaces of $\omega_D|(\theta = \pi/2)$.

5 Unique Determination of ω_D

By symmetry reasons, the *unique* θ-dependent extensions $\hat{\omega}$ of ω are of the form $sin\ \theta\ d\theta$, $sin\ \theta\ d\varphi$, $cos\ \theta\ d\varphi$, and $cos\ \theta\ d\theta$. The first two lead to $\hat{\omega}(\theta = \pi/2) = \frac{i}{2}(d\chi + d\theta)$ or $\frac{i}{2}(d\chi + d\varphi)$ which are different from $\omega_D|(\theta = \pi/2)$, while the fourth one leads to $\hat{\omega} = \frac{i}{2}(d\chi + dsin\ \theta) = \frac{i}{2}d\chi'$, with $\chi' = \chi + sin\ \theta$, which is the same as ω. So, the unique θ-dependent extension of ω is the restriction to $S^3 \setminus T^2$ of the Dirac connection:

$$\hat{\omega}(\theta) = \omega_D|(\theta). \tag{42}$$

Since ω in $\hat{\xi_D}$ is uniquely determined by A_{A-B} in ξ_{A-B}, its θ-dependent extension $\hat{\omega}$ is unique, and, as previously mentioned, the transition from $\theta \in (0, \pi)$ to $\theta \in [0, \pi]$ is continuous, then A_{A-B} uniquely determines ω_D. This ends the proof of the existence and uniqueness of the D connection from the existence of the A_{A-B} connection.

6 Final Comment

The present note does not claim to prove the physical existence of the Dirac monopole, but only to reinforce this idea by showing that, at the mathematical level, in particular in the context of fiber bundle theory, the Aharonov-Bohm connection, relevant to the physically observed $A - B$ effect, implies the existence and uniqueness of the connection which represents the till now hypotetical Dirac charge.

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