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Global Existence and Asymptotic Behavior for a Two-Phase Model with Magnetic Field in a Bounded Domain

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Abstract In this paper, we study the initial boundary value problem for a two-phase with a magnetic field in a bounded domain $\Omega \subset \mathbb{R}^3$. We mainly use the energy method to obtain the global existence of the strong solution and the decay estimate, when the initial value reaches the equilibrium state in $H^2(\Omega)$. At last, we also obtain large time behavior of the solution.

Keywords: two-phase model; magnetic field; bounded domain; asymptotic behavior; global existence; energy method

1 Introduction

1.1 Background and Motivation

In this paper, we are interested in a two-phase system with magnetic field model the motion of the mixture of the fluid and particles in a smooth bounded domain $\Omega \subset \mathbb{R}^3$. The system as following

$$\begin{cases} n_t + \mathrm{div}(nu) = 0, \\ \rho_t + \mathrm{div}(\rho u) = 0, \\ [(\rho+n)u]_t + \mathrm{div}((\rho+n)u \otimes u) + \nabla P - \mu \Delta u - (\lambda+\mu)\nabla \mathrm{div} u = (\nabla \times H) \times H, \\ H_t - \nabla \times (u \times H) = -\nabla \times (\nu \nabla \times H), \\ \mathrm{div} H = 0, \end{cases} \tag{1.1}$$

with the initial and boundary conditions

$$\begin{cases} (n, \rho, u, H)(x, 0) = (n_0, \rho_0, u_0, H_0)(x), \quad x \in \Omega, \\ u(x,t)|_{\partial\Omega} = 0, \quad H(x,t)|_{\partial\Omega} = 0, \\ \frac{1}{|\Omega|}\int_\Omega n_0(x)\mathrm{d}x = \bar{n}_0 > 0, \\ \frac{1}{|\Omega|}\int_\Omega \rho_0(x)\mathrm{d}x = \bar{\rho}_0 > 0. \end{cases} \tag{1.2}$$

Here, $t \geq 0$ and $x = (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3$. The variables n, $u = (u_1, u_2, u_3)$ and H denote the density of the fluid, the velocity field of the fluid and the magnetic field, respectively. ρ is the density of the particles in the mixture which is related to the probability distribution function $F(t, x, \xi)$ in the macroscopic description through the relation

$$\rho(x,t) = \int_{\mathbb{R}^3} F(x, t, \xi)\mathrm{d}\xi.$$

$P = P(\rho, n)$ is pressure satisfying

$$P = \rho^\alpha + n^\gamma, \tag{1.3}$$

for $\alpha \geq 1$ and $\gamma \geq 1$. The viscosity coefficients μ and λ satisfy

$$\mu > 0, \quad 3\lambda + 2\mu \geq 0. \tag{1.4}$$

The constant $\nu > 0$ is the resistivity coefficient.

System (1.1) is derived from the Vlasov-Fokker-Planck/compressible magnetohydrodynamics equations by Wen and Zhu in [26] , taking the form of

$$\begin{cases} (n_\epsilon)_t + \mathrm{div}(n_\epsilon u_\epsilon) = 0, \\ (n_\epsilon u_\epsilon)_t + \mathrm{div}(n_\epsilon u_\epsilon \otimes u_\epsilon) - \mu \Delta u_\epsilon - (\mu+\lambda)\nabla \mathrm{div} u_\epsilon + \nabla P_\epsilon \\ = (\nabla \times H_\epsilon) \times H_\epsilon + \frac{1}{\epsilon}\int_{\mathbb{R}^3}(v-u_\epsilon)F_\epsilon \mathrm{d}v, \\ (H_\epsilon)_t - \nabla \times (u_\epsilon \times H_\epsilon) = -\nabla \times (\nu \nabla \times H_\epsilon), \\ \mathrm{div} H_\epsilon = 0, \\ (F_\epsilon)_t + v\cdot \nabla F_\epsilon + \frac{1}{\epsilon}\mathrm{div}[(u_\epsilon - v)F_\epsilon - \nabla F_\epsilon] = 0, \end{cases} \tag{1.5}$$

Carrillo and Goudon in [3] applying ideas when the scaling limit $\epsilon \to 0^+$. The system (1.5) describes the motions of the mixture of fluid and particles in magnetic field. This type of fluid-particle interaction model can be used to engineering, medicine, geophysics and astrophysics.

In the absence of a magnetic field, Wu and Zhang in [25] investigated the viscous two-phase model and get the global existence and asympotic behavior of strong solution in a bounded domain with no-slip boundary. For the incompressible Vlasov-Navier-Stokes system without the Fokker-Planck term, Boudin, Desvillettes, et al. obtained the global existence of weak solutions on periodic domain in [2] and by Yu on bounded domain in [27]. To the free boundary problem, When the liquid is incompressible and the gas is polytropic, the existence, uniqueness, regularity, asymptotic behavior and decay rate estimates of (weak or classical) solutions have been studied in [6,8,9,18]. Evje-Flåatten [7] obtained the global existence of weak solutions, when the both of two fluids are compressible. As a generalization of the results in [7] to high dimensions, when the initial energy is small and the initial density is bounded far away from the vacuum, Yao, Zhang and Zhu proved the existence of the global solution to the 2D model in [28]. For the Cauchy problem of the liquid-gas two phase flow model:

$$\begin{cases} m_t + \mathrm{div}(mu) = 0, \\ n_t + \mathrm{div}(nu) = 0, \\ (mu)_t + \mathrm{div}(mu \otimes u) + \nabla P(m,n) = \mu \Delta u + (\lambda+\mu)\nabla \mathrm{div} u, \end{cases} \tag{1.6}$$

When viscosity is present, Yao, Zhang and Zhu studied the existence and asymptotic properties of the system (1.6) with initial values, which is widely used in the petroleum industry [28]. Evje and Karlsenre explore the well and pipe flow, the introduction of a novel two-phase variant of the potential energy function needed for obtaining fundamental a priori estimates and derive the existence of weak solutions. This method plays an important role in the single-phase Navier-Stokes equations[7]. Hoff demonstrated the weak solution of the compressible Navier-Stokes equation that the initial velocity is small on L^2 and bounded on L^{2n} as the initial density approaches a constant on L^2 and L^∞, we can refer [12,13,14] for more detail. When the initial is vacuum and small enough, Guo, Yang and Yao obtained the existence of the global strong solution [11,16]. When both the initial liquid and gas masses connect to vacuum continuously, Zhu et al. studied the system (1.6) with constant viscosity coefficient, and got the global existence of weak solution and the uniqueness of the weak solution by the line method. At a distance from the vacuum and a small initial energy, the global existence of system (1.6) is obtained by Yao, Zhang and Zhu. The global classical solutions were first obtained by Matsumura and Nishida [19,20,21] for initial data close to a non-vacuum equilibrium in $H^2(\mathbb{R}^3)$. In particular, the theory requires that the solution has small oscillations from a uniform non-vacuum state so that the density is strictly away from the vacuum and the gradient of the density remains bounded uniformly in time. Later, Hoff and Huang et al.[15] studied the problem for discontinuous initial data. When there is an external force in the system, Zhang and Fang prove the global existence of weak solutions for the 2D compressible Navier-Stokes equations with a density-dependent viscosity coefficient ($\lambda = \lambda(\rho)$) and show that the viscosity coefficient μ plays a key role in the Navier-Stokes equations, when the condition of μ=constant is constant will induce a singularity of the system at vacuum [29].

In recent years, the study of two-phase flow models with magnetic fields in bounded domain is becoming increasingly popular [4,10,17,22,23,30]. The free boundary value problem for two-phase liquid-gas model with mass-dependent viscosity coefficient when both the initial liquid and gas masses connect to vacuum with a discontinuity is studid by Zhu and Yao in [31]. In this paper, we use traditional energy estimation methods, but the boundary conditions are not smooth enough, so we refer to [24] deal with the spatial derivatives and using the local geodesic polar coordinate.

1.2 Main Results

We first rewrite system (1.1) in a more suitable form. Let

$$c := \rho^\alpha - n^\gamma, \qquad m := \rho + n. \tag{1.7}$$

By a direct calculation, from (1.3) and $(1.7)_1$, we have

$$\rho = \left(\frac{P+c}{2}\right)^{\frac{1}{\alpha}}, \quad n = \left(\frac{P-c}{2}\right)^{\frac{1}{\gamma}}, \tag{1.8}$$

and

$$\rho^\alpha = \frac{1}{2}(P+c), \quad n^\gamma = \frac{1}{2}(P-c). \tag{1.9}$$

Since

$$\begin{aligned}(\nabla \times H) \times H &= (H \cdot \nabla)H - \frac{1}{2}\nabla(|H|^2),\\ \Delta H &= \nabla \mathrm{div} H - (\nabla \times H) \times H,\\ (\nabla \times u) \times H &= (H \cdot \nabla)u - (u \cdot \nabla)H + u\mathrm{div}H - H\mathrm{div}u,\end{aligned}$$

then the system (1.1) clearly can be written in terms of the variables (c, P, u, H), that is

$$\begin{cases} c_t + u \cdot \nabla c = B_1 \mathrm{div} u,\\ P_t + u \cdot \nabla P = B_2 \mathrm{div} u,\\ m u_t + (mu \cdot \nabla)u + \nabla P = \mu \Delta u + (\lambda + \mu)\nabla \mathrm{div} u + (\nabla \times H) \times H,\\ H_t - \nu \Delta H = (H \cdot \nabla)u - (u \cdot \nabla)H - H\mathrm{div}u, \end{cases} \tag{1.10}$$

with the initial and boundary conditions

$$\begin{cases} (c, P, u, H)(x, 0) = (c_0, P_0, u_0, H_0)(x), \qquad x = (x_1, x_2, x_3) \in \Omega,\\ u(x,t)|_{\partial\Omega} = 0, \quad H(x,t)|_{\partial\Omega} = 0, \quad t \geq 0,\\ \frac{1}{|\Omega|}\int_\Omega P_0 dx = \bar{P}_0, \end{cases} \tag{1.11}$$

where $B_1 = -\left(\frac{\alpha-\gamma}{2}\mathrm{P} + \frac{\alpha+\gamma}{2}\mathrm{c}\right)$, $B_2 = -\left(\frac{\alpha+\gamma}{2}\mathrm{P} + \frac{\alpha-\gamma}{2}\mathrm{c}\right)$ and $\bar{P}_0$ is a positive constant due to (1.3).

Theorem 1.1. *Assume the initial boundary value $(c_0, P_0 - \bar{P}_0, u_0, H_0)$ satifies the compatibility conditions, i.e. $\partial_t^l u(0)\big|_{\partial\Omega} = 0$, $\partial_t^l H(0)\big|_{\partial\Omega} = 0$, $l = 0, 1$. Where $\partial_t^l u(0)\big|_{\partial\Omega} = 0$, $\partial_t^l H(0)\big|_{\partial\Omega} = 0$ show the lth derivative at $t = 0$ of any solution of the system* (1.1)-(1.4)*, as calculated from* (1.1) *to yield an expression in terms of c_0, P_0, u_0, H_0. Then there exists a constant ε_0 such that if*

$$\left\|(c_0, P_0 - \bar{P}_0, u_0, H_0)\right\|_2 \leq \varepsilon_0 \tag{1.12}$$

then the initial boundary value problem (1.1)-(1.4) admits a unique solution (c, P, u, H) globally in the time with $P > 0$, which satifies

$$\begin{aligned} P - \bar{P}, c \in& C^0([0,\infty); H^2(\Omega)) \cap C^1([0,\infty); H^1(\Omega)),\\ u, H \in& C^0([0,\infty); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega)), \end{aligned} \tag{1.13}$$

where

$$\bar{P}(t) = \frac{1}{|\Omega|}\int_{\Omega} P(x,t)dx. \tag{1.14}$$

Moreover, there exist two positive constant C_1, C_2 such that for any $t > 0$, it holds that

$$\begin{aligned} &\left\|(P-\bar{P},u,H)(t)\right\|_2^2 + \left\|\partial_t(P-\bar{P},u,H)(t)\right\|_{L^2}^2 \\ \leq &C_1\left\|(P_0-\bar{P}_0,u_0,H_0)(t)\right\|_2^2 \exp\{-c_2 t\}, \end{aligned} \tag{1.15}$$

$$\begin{aligned} &\left\|(P-\bar{P},u,H)(t)\right\|_2^2 + \int_0^t \left\|(P-\bar{P})(\tau)\right\|_2^2 + \|(u,H)(\tau)\|_3^2 d\tau \\ \leq &C_1\left\|(P_0-\bar{P}_0,u_0,H_0)(t)\right\|_2^2, \end{aligned} \tag{1.16}$$

$$\|c\|_2 \leq C_1\exp\left\{C_1\left\|(P_0-\bar{P}_0,u_0,H_0)\right\|_2\right\}\left(\|u_0\|_2 + \left\|(P_0-\bar{P}_0,u_0,H_0)\right\|_2^2\right). \tag{1.17}$$

Finally, $\lim_{t\to\infty}\bar{P}(t)$ exists and let $\lim_{t\to\infty}\bar{P}(t) = \tilde{P}$, the following convergence rate holds

$$|\tilde{P} - \bar{P}(t)| \leq C_0\|(c_0, P_0-\bar{P},u_0,H_0)\|_{H^2}^2 \exp\{-\eta_0 t\}. \tag{1.18}$$

Notations. Before we start the proof, let's introduce some notations in this article. C denotes the generic positive constant depending only on the initial data but independent of time t. In Sobolev spaces, the norms $H^m(\mathbb{R}^3)$ and $W^{m,p}(\mathbb{R}^3)$ are denoted respectively by $\|\cdot\|_{H^m}$ and $\|\cdot\|_{W^{m,p}}$, for $m \geq 0$ and $p \geq 1$. Especially, when $m = 0$, we will simply use $\|\cdot\|_{L^2}$ and $\|\cdot\|_{L^p}$. Finally, $\nabla = (\partial_1, \partial_2, \partial_3)$, $\partial_i = \partial_{x_i}(i = 1, 2, 3)$ and for any integer $l \geq 0$, $\nabla^l f$ denotes all derivatives of order l of the function f.

The rest of this paper is organized in the following way. In section 2, we will show some useful inequalities. In Section 3, we obtained some *a priori* estimates and hence the global existence by the energy estimate method and low-high-frequency decomposition. At the same time, we finished the proof of Theorem 1.1. As a by-product, we also get the time decay estimates of the solutions.

2 Preliminaries

In this section, we first introduce some Sobolev's inequalities that will be used frequently in later articles.

Lemma 2.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and $f \in H^2$. It holds that*

$$\begin{aligned} &(i)\ \|f\|_{L^\infty} \leq C\|f\|_2, \\ &(ii)\ \|f\|_{L^p} \leq C\|f\|_1,\ 2 \leq p \leq 6, \end{aligned} \tag{2.1}$$

for some contants $C > 0$ depending only on Ω.

The classical energy estimates don't work in estimating the spatial derivatives of the solutions with the slip boundary condition. In order to get the estimates on the tangential derivatives of the solutions (P, u, H), we refer to the following lemmas on the stationary Stokes equations, c.f. [21].

Lemma 2.2. *[25] Let Ω be any bounded domain in $\mathbb{R}^3$ with smooth boundary. Consider the problem*

$$\begin{cases} -\mu\Delta u + \nabla P = g, \\ \mathrm{div} u = f, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2.2}$$

where $f \in H^{k+1}(\Omega), g \in H^k (k \geq 0)$. Then the above problem has a solution $(P, u) \in H^{k+1} \times (H^{k+2} \cap H_0^1)$ which is unique modulo a constant of integration for P. Moreover, this solution satifies

$$\|u\|_{k+2}^2 + \|\nabla P\|_k^2 \leq C\left(\|f\|_{k+1}^2 + \|g\|_k^2\right). \tag{2.3}$$

Lemma 2.3. *[5] Assume U is a bounded, open subset of $(\mathbb{R}^n)$. Suppose $u \in W_0^{1,p}(U)$ for some $1 \leq P < n$. Then we have the estimate*

$$\|u\|_{L^q(U)} \leq C\|Du\|_{L^P(U)} \tag{2.4}$$

for each $q \in [1, p^*]$, the constant C depending only on p,q,n and U.

3 Global Existence

In this section, we will prove the global existence and large time behavior of the solution with the small initial data. The global existence of solutions to the initial boundary value problem (1.10)-(1.11) will be established from the combination of the local existence result with some *a priori* global estimates by employing the standard continuity arguments.

Proposition 3.1. *(Local existence). Let* $(c_0, P_0, u_0, H_0) \in H^2(\Omega)$ *such that*

$$\inf_{x\in\bar{\Omega}}\{P_0(x)\} > 0, \quad \partial_t^l u_0\big|_{\partial\Omega} = 0, \quad \partial_t^l H_0\big|_{\partial\Omega} = 0, \quad l = 0, 1.$$

Then there exists a positive constant C *and* $T_0 > 0$*, such that the initial value problem* (1.1)-(1.4) *has a unique solution* $(c, P, u, H) \in C([0,T]; H^2(\Omega))$ *satisfying*

$$\inf_{t\in[0,T],x\in\bar{\Omega}}\{P(t,x)\} > 0, \quad c_t, P_t \in C([0,T]; H^1(\Omega)),$$
$$u, H \in L^2([0,T]; H^3(\Omega)), \quad u_t, H_t \in C([0,T]; L^2(\Omega)).$$

Furthermore, the following estimates hold,

$$\|c(t)\|_2 + \|P(t)\|_2 + \|u(t)\|_2 + \|H(t)\|_2 \le C(\|c_0\|_2 + \|P_0\|_2 + \|u_0\|_2 + \|H_0\|_2).$$

Remark 3.1. *The proof can be done by using the standard iteration arguments.*

In what follows, we will establish some *a priori* estimates of the solution (P, u, c, H). Firstly, we make the *a priori* assumption that

$$\left\|(P-\bar{P}, u, c, H)(t)\right\|_2 + |\bar{P}(t) - \bar{P}_0| \le \varepsilon, \quad \text{for any} \quad t \ge 0, \tag{3.1}$$

where $\delta > 0$ is sufficiently small. By the Sobolev inequality, we have

$$\frac{1}{2}\bar{P}_0 \le P(t) \le 2\bar{P}_0, \quad \frac{1}{C} \le m(t) \le C, \quad \text{for any} \quad t \ge 0. \tag{3.2}$$

This will be often used in the rest of paper. In order to derive *a priori* estimate, We first use the energy estimation method to estimate the lower derivative of (P, u, H).

Lemma 3.1. *Under the conditions of Theorem 1.1 and* (3.1)*, there exists a positive constant* C *such that for any* $t \ge 0$*, it holds*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega m\,|u|^2 + H^2 + \frac{(P-\bar{P})^2}{\bar{B}_2}dx + \mu\int_\Omega |\nabla u|^2\,dx + (\mu+\lambda)\int_\Omega |\mathrm{div}u|^2\,dx\\ \le& C\varepsilon(\|\nabla P\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2).\end{aligned} \tag{3.3}$$

Proof. Multiplying $(1.10)_3$ and $(1.10)_4$ by u and H respectively, then summing up and integrating on Ω, using integration by parts, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega m\,|u|^2 + H^2 dx + \mu\int_\Omega |\nabla u|^2\,dx + (\mu+\lambda)\int_\Omega |\mathrm{div}u|^2\,dx\\ &+\nu\int_\Omega |\nabla H|^2\,dx + \int_\Omega (\nabla\times H)\times H\cdot u dx\\ =&\int_\Omega ((H\cdot\nabla)u - (u\cdot\nabla)H - H\mathrm{div}u)\cdot H dx.\end{aligned} \tag{3.4}$$

In order to get the estimate of P, we shall deduce the equation of $\bar{P}$. By integrating $(1.10)_2$ over Ω gives

$$\begin{aligned}&\int_\Omega P_t dx + \int_\Omega u\cdot\nabla P dx + \int_\Omega B_2\mathrm{div}u dx\\ =&\int_\Omega P_t dx + \int_\Omega u\cdot\nabla P dx - \int_\Omega u\cdot\nabla B_2 dx = 0.\end{aligned} \tag{3.5}$$

Therefore,

$$\int_{\Omega} P_t dx = \int_{\Omega} u \cdot \nabla(B_2 - P)dx. \tag{3.6}$$

according to (1.14), we have

$$\bar{P}_t(t) = \frac{1}{|\Omega|}\int_{\Omega} P_t(x,t)dx = \frac{1}{|\Omega|}\int_{\Omega} u \cdot \nabla(B_2 - P)dx. \tag{3.7}$$

So,

$$\begin{aligned}\left|\bar{P}_t\right| &\leq C\left\|u\right\|_{L^2}\left(\left\|\nabla B_2\right\|_{L^2} + \left\|\nabla P\right\|_{L^2}\right)\\ &\leq C\left\|u\right\|_{L^2}\left(\left\|\nabla c\right\|_{L^2} + \left\|\nabla P\right\|_{L^2}\right).\end{aligned} \tag{3.8}$$

Equation $(1.10)_2$ is obtained by deformation

$$\frac{(P-\bar{P})_t}{\bar{B}_2} + \mathrm{div}u + \frac{\bar{P}_t + (B_2 - \bar{B}_2)\mathrm{div}u + u\cdot\nabla P}{\bar{B}_2} = 0. \tag{3.9}$$

Multiplying the above equality by $P - \bar{P}$ and integrating over Ω gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\Omega}\frac{(P-\bar{P})^2}{\bar{B}_2}dx + \int_{\Omega}\mathrm{divu}(P-\bar{P})dx\\ =&-\frac{1}{2}\frac{d}{dt}\int_{\Omega}\frac{(P-\bar{P})^2(\bar{B}_2)_{\bar{P}}\bar{P}_t}{\bar{B}_2^2}dx + \int_{\Omega}\frac{\bar{P}_t + (B_2-\bar{B}_2)\mathrm{div}u + u\cdot\nabla P}{\bar{B}_2}(P-\bar{P})dx,\end{aligned} \tag{3.10}$$

Adding (3.4) to (3.10), we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\Omega} m\left|u\right|^2 + H^2 + \frac{(P-\bar{P})^2}{\bar{B}_2}dx + \mu\int_{\Omega}\left|\nabla u\right|^2dx + (\mu+\lambda)\int_{\Omega}\left|\mathrm{div}u\right|^2dx\\ =&-\frac{1}{2}\frac{d}{dt}\int_{\Omega}\frac{(P-\bar{P})^2(\bar{B}_2)_{\bar{P}}\bar{P}_t}{\bar{B}_2^2}dx + \int_{\Omega}\frac{\bar{P}_t + u\cdot\nabla P}{\bar{B}_2}(P-\bar{P})dx\\ &+\int_{\Omega}\frac{(B_2-\bar{B}_2)\mathrm{div}u}{\bar{B}_2}(P-\bar{P})dx + \int_{\Omega}(((H\cdot\nabla)u - (u\cdot\nabla)H - H\mathrm{div}u)\cdot H + \nu\left|\nabla H\right|^2)dx\\ =:&I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{3.11}$$

For the first term on the right

$$\begin{aligned}I_1 &\leq \left|\frac{(\bar{B}_2)_{\bar{P}}\bar{P}_t}{\bar{B}_2}\right|\left\|P-\bar{P}\right\|_{L^2}^2\\ &\leq C\left\|u\right\|_{L^2}\left(\left\|\nabla c\right\|_{L^2} + \left\|\nabla P\right\|_{L^2}\right)\left\|P-\bar{P}\right\|_{L^2}^2\\ &\leq C\varepsilon\left\|\nabla P\right\|_{L^2}^2.\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}I_2 &\leq \left|\frac{C}{\bar{B}_2}\right|\left(\left|\bar{P}_t\right|\left\|P-\bar{P}\right\|_{L^2} + \left\|u\right\|_{L^3}\left\|\nabla P\right\|_{L^2}\left\|P-\bar{P}\right\|_{L^6}\right)\\ &\leq C\left(\left\|u\right\|_{L^2}\left(\left\|\nabla c\right\|_{L^2} + \left\|\nabla P\right\|_{L^2}\right)\left\|P-\bar{P}\right\|_{L^2} + \left\|\nabla u\right\|_{L^2}\left\|\nabla P\right\|_{L^2}^2\right)\\ &\leq C\varepsilon\left(\left\|\nabla u\right\|_{L^2}^2 + \left\|\nabla P\right\|_{L^2}^2\right).\end{aligned} \tag{3.13}$$

By the same way,

$$
\begin{aligned}
I_3 &\leq \int_\Omega \left| \frac{(B_2-\bar{B}_2)}{\bar{B}_2} \right| \mathrm{div} u (P-\bar{P}) dx \\
&\leq C \left\| \frac{(B_2-\bar{B}_2)}{\bar{B}_2} \right\|_{L^3} \|\mathrm{div} u\|_{L^2} \left\| P-\bar{P} \right\|_{L^6} \\
&\leq C\varepsilon \|\mathrm{div} u\|_{L^2} \|\nabla P\|_{L^2} \\
&\leq C\varepsilon \left(\|\nabla u\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 \right).
\end{aligned} \tag{3.14}
$$

and the last term can be estimated by

$$
\begin{aligned}
I_4 &\leq \|H\|_{L^3} \|\nabla u\|_{L^2} \|H\|_{L^6} + \|u\|_{L^3} \|\nabla H\|_{L^2} \|H\|_{L^6} + \|H\|_{L^3} \|\mathrm{div} u\|_{L^2} \|H\|_{L^6} \\
&\leq C\varepsilon \left(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \right).
\end{aligned} \tag{3.15}
$$

Plugging (3.12)-(3.15) into (3.11) yields (3.3). □

Next, we give the energy estimate of the time derivative for (P, u, H).

Lemma 3.2. *Under the conditions of Theorem* 1.1 *and* (3.1)*, there exists a positive constant C such that for any $t \geq 0$ it holds*

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega m|u_t|^2 + H_t^2 + \frac{(P_t-\bar{P}_t)^2}{\bar{B}_2} dx + \mu \int_\Omega |\nabla u_t|^2 dx \\
&+ (\mu+\lambda)\int_\Omega |\mathrm{div} u_t|^2 dx + \nu \int_\Omega |\nabla H_t|^2 dx \\
\leq & C\varepsilon(\|\nabla u\|_1^2 + \|\nabla u_t\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2).
\end{aligned} \tag{3.16}
$$

Proof. Differentiating $(1.10)_3 - (1.10)_4$ and (3.9) with respect to t, then multiplying the result by u_t, H_t and $(P-\bar{P})_t$ respectively, then summing up and integrating on Ω, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega m|u_t|^2 + H_t^2 + \frac{(P_t-\bar{P}_t)^2}{\bar{B}_2} dx + \mu \int_\Omega |\nabla u_t|^2 dx \\
&+ (\mu+\lambda)\int_\Omega |\mathrm{div} u_t|^2 dx + \nu \int_\Omega |\nabla H_t|^2 dx \\
=& -\frac{1}{2}\int_\Omega \left(m_t |u_t|^2 - \frac{(P_t-\bar{P}_t)^2(\bar{B}_2)_{\bar{P}}\bar{P}_t}{(\bar{B}_2)^2} \right) dx - \int_\Omega u_t (mu\cdot\nabla u)_t dx \\
&+ \int_\Omega u_t((\nabla\times H)\times H)_t dx - \int_\Omega \left(\frac{\bar{P}_t + (B_2-\bar{B}_2)\mathrm{div} u + u\cdot\nabla P}{\bar{B}_2} \right)_t (P_t-\bar{P}_t) dx \\
&+ \int_\Omega \left[(H\cdot\nabla u)_t H_t - (u\cdot\nabla H_t) H_t - (H \mathrm{div} u)_t H_t \right] dx \\
=:& J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned} \tag{3.17}
$$

We use the boundary conditions $u_t|_{\partial\Omega} = 0,\ H_t|_{\partial\Omega} = 0$. According to (3.1), (3.2) and Lemma 2.1, Hölder's inequality and Poincar*é* inequality, we obtain

$$
\begin{aligned}
J_1 &= \left| \int_\Omega \mathrm{div}(mu)|u_t|^2 + \frac{(P_t-\bar{P}_t)^2(\bar{B}_2)_{\bar{P}}\bar{P}_t}{(\bar{B}_2)^2} \right| dx \\
&= \left| \int_\Omega 2mu\nabla u_t u_t - \frac{(P_t-\bar{P}_t)^2(\bar{B}_2)_{\bar{P}}\bar{P}_t}{(\bar{B}_2)^2} \right| dx \\
&\leq C\|m\|_{L^\infty} \|u\|_{L^3} \|\nabla u_t\|_{L^2} \|u_t\|_{L^6} + \left| \frac{(\bar{B}_2)_{\bar{P}}\bar{P}_t}{(\bar{B}_2)^2} \right| \left(\|P_t\|_{L^2}^2 + \left\|\bar{P}_t\right\|_{L^2}^2 \right) \\
&\leq C\varepsilon \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2}^2 + C\varepsilon \|\nabla u\|_1^2 \\
&\leq C\varepsilon(\|\nabla u\|_1^2 + \|\nabla u_t\|_{L^2}^2)
\end{aligned} \tag{3.18}
$$

By (3.1) and $(1.10)_2$ yield

$$\begin{aligned}\|P_t\|_{L^2} &\le C(\|u\cdot\nabla P\|_{L^2}+\|\nabla u\|_{L^2})\\ &\le C(\|u\|_{L^\infty}\|\nabla P\|_{L^2}+\|\nabla u\|_{L^2})\\ &\le C\|\nabla u\|_1 .\end{aligned}\tag{3.19}$$

and

$$\begin{aligned}J_2=&\int_\Omega u_t(mu\cdot\nabla u)_t dx\\ =&\int_\Omega (m_t u\cdot\nabla u u_t+mu_t\nabla u u_t+mu\cdot\nabla u_t u_t)dx\\ =&\int_\Omega (-(\nabla mu+m\mathrm{div}u)u\cdot\nabla u u_t+mu_t\nabla u u_t+mu\cdot\nabla u_t u_t)dx\\ \le& C\Big(\|\nabla m\|_{L^3}\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}\|u_t\|_{L^6}+\|m\|_{L^\infty}\|\mathrm{div}u\|_{L^2}\|u\|_{L^\infty}\|\nabla u\|_{L^3}\|u_t\|_{L^6}\\ &+\|m\|_{L^\infty}\|\nabla u\|_{L^3}\|u_t\|_{L^3}^2+\|m\|_{L^\infty}\|u\|_{L^3}\|\nabla u_t\|_{L^2}\|u_t\|_{L^6}\Big)\\ \le& C\varepsilon(\|\nabla u\|_{L^2}^2+\|\nabla u_t\|_{L^2}^2),\end{aligned}\tag{3.20}$$

where is using

$$\|\nabla m\|_{L^3}\le\|\nabla m\|_1\le C\left(\|\nabla P\|_1+\|\nabla c\|_1\right).\tag{3.21}$$

Similarly,

$$\begin{aligned}J_3&=\int_\Omega u_t((\nabla\times H)\times H)_t dx\\ &\le\int_\Omega u_t(H_t\cdot\nabla H+H\cdot\nabla H_t)dx\\ &\le\|u_t\|_{L^6}\|H_t\|_{L^6}\|\nabla H\|_{L^3}^2+\|u_t\|_{L^6}\|H\|_{L^3}\|\nabla H_t\|_{L^2}\\ &\le\|u_t\|_{L^6}\|H_t\|_{L^6}\|\nabla H\|_{L^3}^2+\|u_t\|_{L^6}\|H\|_{L^3}\|\nabla H_t\|_{L^2}\\ &\le C\varepsilon(\|\nabla H_t\|_{L^2}^2+\|\nabla u_t\|_{L^2}^2).\end{aligned}\tag{3.22}$$

Then, combing (3.1), $(1.10)_2$, (3.7) and (3.19)

$$\begin{aligned}\left|\bar P_{tt}\right|&\le\frac{1}{|\Omega|}\int_\Omega\left(u_t\cdot\nabla(B_2-P)+u\cdot\nabla(B_2-P)_t\right)dx\\ &\le C\left[\|u_t\|_{L^2}\left(\|\nabla P\|_{L^2}+\|\nabla c\|_{L^2}\right)+\|\nabla u\|_{L^2}\left(\|P_t\|_{L^2}+\|c_t\|_{L^2}\right)\right]\\ &\le C\varepsilon\left(\|u_t\|_{L^2}+\|\nabla u\|_{L^2}+\|c_t\|_{L^2}\right)\\ &\le C\varepsilon\left(\|u_t\|_{L^2}+\|\nabla u\|_{L^2}\right),\end{aligned}\tag{3.23}$$

Finally,

$$\begin{aligned}J_4=&\int_\Omega\left(\frac{\bar P_t+(B_2-\bar B_2)\mathrm{div}u+u\cdot\nabla P}{\bar B_2}\right)_t(P_t-\bar P_t)dx\\ \le&\left|\int_\Omega\frac{\left\{\bar P_{tt}+\left[(B_2)_P P_t+(B_2)_c c_t-(\bar B_2)_{\bar P}\bar P_t\right]\mathrm{div}u-(B_2-\bar B_2)\mathrm{div}u_t+u\cdot\nabla P\right\}(P_t-\bar P_t)}{\bar B_2}dx\right|\\ &+\left|\int_\Omega\frac{(u_t\cdot\nabla P+u\cdot\nabla P_t)(P_t-\bar P_t)}{\bar B_2}dx\right|\\ &+\left|\int_\Omega\frac{\left\{\bar P_t+(B_2-\bar B_2)\mathrm{div}u+u\cdot\nabla P\right\}(\bar B_2)_{\bar P}\bar P_t(P_t-\bar P_t)}{(\bar B_2)^2}dx\right|\\ \le& C\varepsilon(\|\nabla u\|_1^2+\|\nabla u_t\|_{L^2}^2).\end{aligned}\tag{3.24}$$

Substituting (3.18), (3.20), (3.22), (3.24) into (3.17) and we get (3.16). We complete the prove of Lemma 3.2. □

Because the boundary conditions are not smooth, the classical energy estimates can not be applied directly to spatial derivatives. To solve the difficult, we need to localize $\partial\Omega$ when estimate the boundary solution. Detail description, we will modify the standard technique developed in [20] that involves separating the estimates of solution into that over the region away from the boundary and near the boundary. Let χ_0 be an arbitrary but fixed function in $C_0^\infty(\Omega)$. Then we have the following energy estimates on the region away from the boundary.

Lemma 3.3. *Under the conditions of Theorem 1.1 and (3.1), there exists a positive constant C such that for any $t \geq 0$, it holds*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega m\left|\nabla u\chi_0\right|^2+\frac{\left|\nabla P\chi_0\right|^2}{\bar{B}_2}+\upsilon\left|\nabla H\chi_0\right|^2dx\\
&+\mu\int_\Omega\left|\nabla^2u\chi_0\right|^2dx+(\mu+\lambda)\int_\Omega\left|\nabla\mathrm{div}u\chi_0\right|^2dx+\int_\Omega\left|\nabla^2H\chi_0\right|^2dx\\
\leq&C\varepsilon\left(\|\nabla u\|_1^2+\|\nabla H\|_1^2+\|\nabla P\|_1^2+\|\nabla u_t\|_{L^2}^2\right)\\
&+C\|\nabla u\|_{L^2}\left(\left\|\nabla^2u\right\|_{L^2}+\|\nabla P\|_{L^2}\right)+\|\nabla H\|_{L^2}\left\|\nabla^2H\right\|_{L^2}.
\end{aligned}\tag{3.25}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega m\left|\nabla^2u\chi_0\right|^2+\frac{\left|\nabla^2P\chi_0\right|^2}{\bar{B}_2}+\upsilon\left|\nabla^2H\chi_0\right|^2dx\\
&+\mu\int_\Omega\left|\nabla^3u\chi_0\right|^2dx+(\mu+\lambda)\int_\Omega\left|\nabla^2\mathrm{div}u\chi_0\right|^2dx+\int_\Omega\left|\nabla^3H\chi_0\right|^2dx\\
\leq&C\varepsilon\left(\|\nabla u\|_2^2+\|\nabla H\|_2^2+\|\nabla P\|_1^2+\|\nabla u_t\|_{L^2}^2\right)\\
&+C\left\|\nabla^2u\right\|_{L^2}\left(\left\|\nabla^3u\right\|_{L^2}+\left\|\nabla^2P\right\|_{L^2}\right)+\left\|\nabla^2H\right\|_{L^2}\left\|\nabla^3H\right\|_{L^2}.
\end{aligned}\tag{3.26}$$

Proof. Differentiating $(1.10)_3$, $(1.10)_4$ and (3.9) with respect to x_i, multiplying the resulting equations by $u_{x_i}\chi_0^2, H_{x_i}\chi_0^2, P_{x_i}\chi_0^2$ respectively, then summing up and integrating on Ω, using integration by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega m\left|u_{x_i}\chi_0\right|^2+\frac{\left|P_{x_i}\chi_0\right|^2}{\bar{B}_2}+\upsilon\left|H_{x_i}\chi_0\right|^2dx\\
&+\mu\int_\Omega\left|\nabla u_{x_i}\chi_0\right|^2dx+(\mu+\lambda)\int_\Omega\left|\mathrm{div}u_{x_i}\chi_0\right|^2dx+\int_\Omega\left|\nabla H_{x_i}\chi_0\right|^2dx\\
=&\frac{1}{2}\int_\Omega m_t\left|u_{x_i}\chi_0\right|^2-\frac{\left|P_{x_i}\chi_0\right|^2(\bar{B}_2)_{\bar{P}}\bar{P}_t}{\bar{B}_2^2}dx-\int_\Omega m_{x_i}u_t+(mu\cdot\nabla u)_{x_i}u_{x_i}\chi_0^2dx\\
&-\int_\Omega(\nabla\times H)\times H\nabla u_{x_i}\chi_0^2dx-\int_\Omega(\nabla\times H)\times Hu_{x_i}\nabla\chi_0^2dx\\
&-\mu\int_\Omega u_{x_i}\nabla u_{x_i}\nabla\chi_0^2dx-(\mu+\lambda)\int_\Omega\mathrm{div}u_{x_i}u_{x_i}\nabla\chi_0^2dx\\
&+\int_\Omega P_{x_i}u_{x_i}\nabla\chi_0^2dx-\upsilon\int_\Omega\nabla^2H\cdot H_{x_i}\nabla\chi_0^2dx\\
&+\int_\Omega\nabla\left[(H\cdot\nabla)u-(u\cdot\nabla)H-H\mathrm{div}u\right]H_{x_i}\chi_0^2dx\\
\leq&C\varepsilon\left(\|\nabla u\|_1^2+\|\nabla H\|_1^2+\|\nabla P\|_1^2+\|\nabla u_t\|_{L^2}^2\right)\\
&+C\|\nabla u\|_{L^2}\left(\left\|\nabla^2u\right\|_{L^2}+\|\nabla P\|_{L^2}\right)+\|\nabla H\|_{L^2}\left\|\nabla^2H\right\|_{L^2}.
\end{aligned}\tag{3.27}$$

So, the Lemma 3.3 can be finished. □

Now, let us estimates near the boundary. Similar to refer [20], we need a more argument using the trick of estimating the tangential derivatives and the normal derivatives separately. We choose a finite number of bounded open sets $\{O_j\}_{j=1}^N$ in $\mathbb{R}^3$, such that $\partial\Omega \subset \cup_{j=1}^N O_j$. In each open set O_j, we choose the local coordinates $y = (y_1, y_2, y_3)$ as follows:
1. The surface $\mathrm{O_j} \cap \partial\Omega$ is the image of a smooth vector function $z^j(y_1, y_2) = (z_1^j, z_2^j, z_3^j)(y_1, y_2)$ (e.g. take the local geodesic polar coordinate), satifying

$$\left|z_{y_1}^j\right| = 1,\ z_{y_1}^j \cdot z_{y_2}^j = 0\ and\ \left|z_{y_2}^j\right| \geq \delta > 0 \tag{3.28}$$

Where δ is a positive constant independent of $1 \leq j \leq N$.
2. For any $x = (x_1, x_2, x_3) \in O_j$ is represented by

$$x_i = \Psi_i(y) = y_3\eta_i^j(z^j(y_1, y_2) + z_i^j(y_1, y_2)),\ for\ i = 1, 2, 3 \tag{3.29}$$

where $\eta^j(y_1, y_2) = (\eta_1^j, \eta_2^j, \eta_3^j)(z^j(y_1, y_2))$ represents the internal unit normal vetor at the point $z^j(y_1, y_2)$ of the surface $\partial\Omega$. In this paper, We omit the subscript j in what follows for the simplicity of presentation. For $k = 1, 2$, we define the unit vectors

$$e_1 = z_{y_1}\ and\ e_2 = \frac{z_{y_2}}{|z_{y_2}|},$$

Then Frenet-Serret's formal gives that there exist smooth functions $(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2)$ of (y_1, y_2) satifying

$$\frac{\partial}{\partial y_1}\begin{pmatrix} e_1 \\ e_2 \\ \eta \end{pmatrix}^i = \begin{pmatrix} 0 & -\gamma_1 & -\alpha_1 \\ \gamma_1 & 0 & -\beta_1 \\ \alpha_1 & \beta_1 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ \eta \end{pmatrix}^i,$$

$$\frac{\partial}{\partial y_1}\begin{pmatrix} e_1 \\ e_2 \\ \eta \end{pmatrix}^i = \begin{pmatrix} 0 & -\gamma_2 & -\alpha_2 \\ \gamma_2 & 0 & -\beta_2 \\ \alpha_2 & \beta_2 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ \eta \end{pmatrix}^i,$$

where e_m^i denote the i-th component of e_m. An elementary calculation shows that the Jacobian J of the transform (3.29) is

$$J = \Psi_{y_1} \times \Psi_{y_2} \cdot \eta = |z_{y_2}| + (\alpha_1|z_{y_2}| + \beta_2)y_3 + (\alpha_1\beta_2 - \alpha_2\beta_1)y_3^2 \tag{3.30}$$

By (3.30),we have the transform (3.28) is regular by choosing y_3 so small that $J \geq \frac{\delta}{2}$ for some positive δ. Therefore, the inverse function of $\Psi(y) := (\Psi_1, \Psi_2, \Psi_3)(y)$ exists, and we use $y = \Psi^{-1}(x)$ denote it. Using a straightforward calculation, $(y_1, y_2, y_3)_{x_i}$ can be expressed by

$$\begin{cases} \partial_{x_i} y_1 = \frac{1}{J}(\Psi_{y_2} \times \Psi_{y_3})_i = \frac{1}{J}(Ae_i^1 + Be_i^2) =: r_{1i}, \\ \partial_{x_i} y_2 = \frac{1}{J}(\Psi_{y_3} \times \Psi_{y_1})_i = \frac{1}{J}(Ce_i^1 + De_i^2) =: r_{2i}, \\ \partial_{x_i} y_3 = \frac{1}{J}(\Psi_{y_1} \times \Psi_{y_2})_i = \eta_i =: r_{1i}, \end{cases} \tag{3.31}$$

Where $A = |z_{y_2}| + \beta_2 y_3,\ B = -y_3\alpha_2,\ C = -\beta_1 y_3,\ D = 1 + \alpha_1 y_3$, and $J = AD - BC \geq \frac{\delta}{2}$. It's easy to find out, (3.31) gives

$$\sum_{i=1}^3 a_{3i}^2 = |n|^2 = 1, r_{1i}r_{3i} = r_{2i}r_{3i} = 0, J^2 = (AC + BD)^2 - (A^2 + B^2)(C^2 + D^2) \tag{3.32}$$

and

$$\partial_{x_i} = r_{ki}\partial_{yk} \tag{3.33}$$

where we have used the Einstein convention of summing over repeated indices. Therefore, in each O_j, $(1.10)_2 - (1.10)_3$ can be rewritten in the local coordinates (y_1, y_2, y_3) as follows:

$$E^p := \frac{dP}{dt} + \frac{\bar{B}_2}{J}\left[(Ae_1 + Be_2) \cdot u_{y_1} + (Ce_1 + De_2) \cdot u_{y_2} + J\eta \cdot u_{y_3}\right] = g,$$

$$
\begin{aligned}
E^u :=&mu_t - \frac{\mu}{J^2}\left[\left(A^2+B^2\right)u_{y_1y_1} + 2(AC+BD)u_{y_1y_2} + (C^2+D^2)u_{y_2y_2}\right.\\
&\left.+\ J^2 u_{y_3y_3}\right] + one\ order\ terms\ of\ u + \frac{1}{J}(Ae_1+Be_2)\left[\frac{\mu+\lambda}{\bar{B}_2}\frac{dP}{dt}+P\right]_{y_1}\\
&+\frac{1}{J}(Ce_1+De_2)\left[\frac{\mu+\lambda}{\bar{B}_2}\frac{dP}{dt}+P\right]_{y_2} + \eta\left[\frac{\mu+\lambda}{\bar{B}_2}\frac{dP}{dt}+P\right]_{y_3} = h,
\end{aligned}
$$

where

$$
\begin{aligned}
&\frac{d}{dt} = \partial_t + u\cdot\nabla(denotes\ the\ material\ derivative),\\
&g = -(B_2-\bar{B}_2)\mathrm{div}u,\\
&h = mu\cdot\nabla u + \frac{\mu+\lambda}{\bar{B}_2}\nabla g + (\nabla\times H)\times H.
\end{aligned}
$$

Let us denote the tangential derivatives by $\partial = (\partial_{y_1}, \partial_{y_2})$ and χ_j be arbitrary but fixed function in $C_0^\infty(O_j)$. Obviously, $x_j\partial^k u = 0\ on\ \partial\Omega_j^{-1}$, where $0\le k\le 2$ and $\Omega_j^{-1}(y) := \left\{y|y=\Psi^{-1}(x), x\in\Omega_j = O_j\cap\Omega\right\}$. Estimating the tangential derivatives in the similar way as the above lemma, we have

Lemma 3.4. *Under the conditions of Theorem* 1.1 *and* (3.1)*, there exists a positive constant C such that for any $t\ge 0, 1\le j\le N$, it holds*

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega_j^{-1}} m\left|\partial u\chi_j\right|^2 + \frac{\left|\partial P\chi_j\right|^2}{\bar{B}_2}dy + \int_{\Omega_j^{-1}}\left|\partial\nabla u\chi_j\right|^2dy + \int_{\Omega_j^{-1}}\left|\partial\frac{dP}{dt}\chi_j\right|^2dy\\
\le& C\varepsilon\left(\|\nabla u\|_1^2 + \|\nabla u_t\|_{L^2}^2 + \|\nabla P\|_1^2\right) + C\|\nabla u\|_{L^2}\left(\|\nabla u\|_1 + \|\nabla P\|_{L^2}\right),
\end{aligned}
\tag{3.34}
$$

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega_j^{-1}} m\left|\partial^2 u\chi_j\right|^2 + \frac{\left|\partial^2 P\chi_j\right|^2}{\bar{B}_2}dy + \int_{\Omega_j^{-1}}\left|\partial^2\nabla u\chi_j\right|^2dy + \int_{\Omega_j^{-1}}\left|\partial^2\frac{dP}{dt}\chi_j\right|^2dy\\
\le& C\varepsilon\left(\|\nabla u\|_2^2 + \|\nabla u_t\|_{L^2}^2 + \|\nabla P\|_1^2\right) + C\left\|\nabla^2 u\right\|_{L^2}\left(\|\nabla u\|_2 + \left\|\nabla^2 P\right\|_{L^2}\right).
\end{aligned}
\tag{3.35}
$$

Next, we begin to deduce the estimates of derivatives in the normal directions.

Lemma 3.5. *Under the conditions of Theorem* 1.1 *and* (3.1)*, there exists a positive constant C such that for any $t\ge 0, k+l=1, k,l\ge 0, 1\le j\le N$, it holds*

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega_j^{-1}}\left|P_{y_3}\chi_j\right|^2dy + \int_{\Omega_j^{-1}}\left|\left(\frac{dP}{dt}\right)_{y_3}\chi_j\right|^2dy\\
\le& C\left[\|\nabla u\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \varepsilon\left(\|\nabla P\|_1^2 + \|\nabla u\|_1^2 + \|\nabla H\|_{L^2}^2\right) + \int_{\Omega_j^{-1}}\left|\partial\nabla u\chi_j\right|^2\right]dy,
\end{aligned}
\tag{3.36}
$$

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega_j^{-1}}\left|\partial^k\partial_{y_3}^{l+1}P\chi_j\right|^2dy + \int_{\Omega_j^{-1}}\left|\partial^k\partial_{y_3}^{l+1}\left(\frac{dP}{dt}\right)_{y_3}\chi_j\right|^2dy\\
\le& C\left[\|\nabla u\|_1^2 + \|u_t\|_1^2 + \varepsilon\left(\|\nabla P\|_1^2 + \left\|\nabla^2 u\right\|_1^2 + \|\nabla H\|_{L^2}^2\right) + \int_{\Omega_j^{-1}}\left|\partial^{k+1}\partial_{y_3}^l\nabla u\chi_j\right|^2\right]dy,
\end{aligned}
\tag{3.37}
$$

Proof. First, we using $\partial_{y_3}(\mathrm{E}^P - g) = 0$ and $\eta(\mathrm{E}^u - h) = 0$, that is the following forms:

$$
\begin{aligned}
&\left(\tfrac{dP}{dt}\right)_{y_3} + \tfrac{\bar{B}_2}{J}\left[(Ae_1+Be_2)\cdot u_{y_1y_3} + (Ce_1+De_2)\cdot u_{y_2y_3} + J\eta\cdot u_{y_3y_3}\right]\\
&+one\ order\ terms\ of\ u = g_{y_3} = \left[-(B_2-\bar{B}_2)\mathrm{div}u\right]_{y_3},
\end{aligned}
\tag{3.38}
$$

$$
\begin{aligned}
&\eta mu_t - \tfrac{\mu}{J^2}\left[(A^2+B^2)\eta u_{y_1y_1} + 2(AC+BD)\eta u_{y_1y_2} + (C^2+D^2)\eta u_{y_2y_2} + J^2\eta u_{y_3y_3}\right]\\
&one\ order\ terms\ of\ u + \left[\tfrac{\mu+\lambda}{\bar{B}_2}\tfrac{dP}{dt} + P\right]_{y_3} = \eta h,
\end{aligned}
\tag{3.39}
$$

In order to eliminate $u_{y_3y_3}$ in equation (3.38), we use (3.38) $\times \frac{\mu}{\bar{B}_2}$ + (3.39) yields:

$$\begin{aligned}\frac{2\mu+\lambda}{\bar{B}_2}\left(\frac{dP}{dt}\right)_{y_3}+P_{y_3}=&-\frac{\mu}{J^2}\left[(A^2+B^2)\eta u_{y_1y_1}\right.\\&\left.+2(AC+BD)\eta u_{y_1y_2}+(C^2+D^2)\eta u_{y_2y_2}\right]\\&-\eta m u_t-\frac{\mu}{J}\left[(Ae_1+Be_2)\cdot u_{y_1y_3}+(Ce_1+De_2)\cdot u_{y_2y_3}+J\eta\cdot u_{y_3y_3}\right]\\&+\mathit{one\ order\ terms\ of}\ u+\eta h+\frac{\mu}{\bar{B}_2}g_{y_3}=\Phi.\end{aligned} \tag{3.40}$$

Multiply that by $\chi_j^2\left(\frac{dP}{dt}\right)_{y_3}$ and integrating on Ω, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\Omega_j^{-1}}|P_{y_3}\chi_j|^2\,dy+\frac{2\mu+\lambda}{\bar{B}_2}\int_{\Omega_j^{-1}}\left|\left(\frac{dP}{dt}\right)_{y_3}\chi_j\right|^2dy\\=&\int_{\Omega_j^{-1}}-(u\cdot\nabla P)_{y_3}P_{y_3}\chi_j^2+\left(\frac{dP}{dt}\right)_{y_3}\Phi\chi_j^2dy.\\=:&\,K_1+K_2.\end{aligned} \tag{3.41}$$

Estimate each term at the right end of the above equation,

$$\begin{aligned}K_1\le&\left|\int_{\Omega_j^{-1}}u_{y_3}\cdot\nabla PP_{y_3}\chi_j^2dy\right|+\frac{1}{2}\left|\int_{\Omega_j^{-1}}(P_{y_3})^2\mathrm{div}(u\chi_j^2)dy\right|\\\le&\,C\|\nabla u\|_1\|\nabla P\|_1^2\le C\varepsilon\|\nabla P\|_1^2,\end{aligned} \tag{3.42}$$

and

$$\begin{aligned}K_2\le&\frac{2\mu+\lambda}{2\bar{B}_2}\int_{\Omega_j^{-1}}\left|\left(\frac{dP}{dt}\right)_{y_3}\chi_j\right|^2dy+C\int_{\Omega_j^{-1}}|\Phi\chi_j|^2dy\\\le&\frac{2\mu+\lambda}{2\bar{B}_2}\int_{\Omega_j^{-1}}\left|\left(\frac{dP}{dt}\right)_{y_3}\chi_j\right|^2dy+C\int_{\Omega_j^{-1}}|\partial\nabla u\chi_j|^2dy\\&+C(\|\nabla u\|_{L^2}^2+\|u_t\|_{L^2}^2+\varepsilon\|\nabla u\|_1^2+\|\nabla H\|_{L^2}^2).\end{aligned} \tag{3.43}$$

Substituting (3.42) and (3.43) into (3.41) get

$$\begin{aligned}&\frac{d}{dt}\int_{\Omega_j^{-1}}|P_{y_3}\chi_j|^2\,dy+\frac{2\mu+\lambda}{\bar{B}_2}\int_{\Omega_j^{-1}}\left|\left(\frac{dP}{dt}\right)\chi_j\right|^2dy\\\le&C\left[\|\nabla u\|_{L^2}^2+\|u_t\|_{L^2}^2+\varepsilon(\|\nabla u\|_1^2+\|\nabla P\|_1^2+\|\nabla H\|_{L^2}^2)+\int_{\Omega_j^{-1}}|\partial\nabla u\chi_j|^2dy\right],\end{aligned} \tag{3.44}$$

By the same way, using $\partial^k\partial_{y_3}^l$ to (3.39), multiplying the resulting equations by $\partial^k\partial_{y_3}^{l+1}\left(\frac{dP}{dt}\right)\chi_j^2$, then when $k+l=1$ we have

$$\begin{aligned}&\frac{d}{dt}\int_{\Omega_j^{-1}}\left|\partial^k\partial_{y_3}^{l+1}P\chi_j\right|^2dy+\frac{2\mu+\lambda}{\bar{B}_2}\int_{\Omega_j^{-1}}\left|\partial^k\partial_{y_3}^{l+1}\left(\frac{dP}{dt}\right)_{y_3}\chi_j\right|^2dy\\\le&C\left[\|\nabla u\|_1^2+\|u_t\|_1^2+\varepsilon(\left\|\nabla^2u\right\|_1^2+\|\nabla P\|_1^2+\|\nabla H\|_{L^2}^2)+\int_{\Omega_j^{-1}}|\partial^{k+1}\partial_{y_3}^l\nabla u\chi_j|^2dy\right].\end{aligned} \tag{3.45}$$

□

Lemma 3.6. *Under the conditions of* 1.1 *and* (3.1)*, there exists a positive constant C such that for any* $t \geq 0$*, it holds*

$$\begin{aligned}&\left(\left\|\nabla^2 u\right\|_{L^2}^2+\left\|\nabla^2 H\right\|_{L^2}^2\right)+\|\nabla P\|_{L^2}^2\\ \leq & C\varepsilon\|\nabla H\|_{L^2}^2+C\left(\left\|\frac{dP}{dt}\right\|_1^2+\|u_t\|_{L^2}^2+\left\|\nabla^2 u\right\|_{L^2}^2\|\nabla u\|_1^2+\|\nabla H_t\|_{L^2}^2\right),\end{aligned} \tag{3.46}$$

$$\begin{aligned}&\int_{\Omega_j^{-1}}\left|\partial\nabla^2 u\chi_j\right|^2 dy+\int_{\Omega_j^{-1}}\left|\partial\nabla P\chi_j\right|^2 dy\\ \leq & C\varepsilon\left\|\nabla^2 H\right\|_{L^2}^2+C\int_{\Omega_j^{-1}}\left|\partial\nabla\frac{dP}{dt}\chi_j\right|^2 dy\\ &+C\left(\|\nabla u\|_1^2+\|u_t\|_1^2+\|\nabla P\|_{L^2}^2+\left\|\nabla^3 u\right\|_{L^2}^2\|\nabla u\|_1^2+\|\nabla P\|_{L^2}\left\|\nabla\frac{dP}{dt}\right\|_{L^2}+\|\nabla H_t\|_{L^2}^2\right).\end{aligned} \tag{3.47}$$

Proof. We consider the Stokes problem of equation $(1.1)_2-(1.1)_3$ as following

$$\begin{cases}\operatorname{div}u=-\frac{1}{B_2}\frac{dP}{dt},\\ -\mu\Delta u+\nabla P=(\lambda+\mu)\nabla\operatorname{div}u-(mu_t+mu\cdot\nabla u)+(\nabla\times H)\times H,\\ u|_{\partial\Omega}=0,\end{cases} \tag{3.48}$$

Where applying (2.2) to (3.48), it can get

$$\left\|\nabla^2 u\right\|_{L^2}^2+\|\nabla P\|_{L^2}^2\leq C\varepsilon(\|\nabla u\|_1^2\left\|\nabla^2 u\right\|_{L^2}^2+\|u_t\|_{L^2}^2+\|\nabla H\|_{L^2}^2)+C\left\|\frac{dP}{dt}\right\|_1^2, \tag{3.49}$$

For the equation $(1.1)_4$, we get the higher order energy estimates of H as

$$\left\|\nabla^2 H\right\|_{L^2}^2\leq C\varepsilon\left(\|\nabla u\|_{L^2}^2+\|\nabla H\|_{L^2}^2\right)+C\|\nabla H_t\|_{L^2}^2, \tag{3.50}$$

Connecting (3.48) – (3.50), we get (3.46). Applying $\chi_j\partial$ to equation (3.48), we have

$$\begin{cases}\operatorname{div}(\chi_j\partial u)=-\chi_j\partial\left(\frac{1}{B_2}\frac{dP}{dt}\right)+\nabla\chi_j\partial u,\\ -\mu\Delta(\chi_j\partial u+\nabla(\chi_j\partial P)=-2\mu\nabla\chi_j\nabla(\partial u)-\Delta\chi_j\partial u+\nabla\chi_j\partial P,\\ -(\lambda+\mu)\chi_j\nabla\partial(\frac{dP}{dt})-\chi_j\partial(mu_t+mu\cdot\nabla u)+\chi_j\partial\left((\nabla\times H)\times H\right),\\ \chi_j\partial u|_{\partial\Omega_j^{-1}}=0,\end{cases} \tag{3.51}$$

Using the Lemma 2.2, same as the proof above, we finished (3.47) . ☐

Now, let's start proving Theorem 1.1. We will do it by four steps.

Step1: we first estimate the lower order derivatives for (P,u,H). Suppose D be a fixed but large positive constant. Let $D^3\times((3.3)+(3.16))+D\times((3.25)+(3.34))+(3.36)$, we can see

$$\begin{aligned}&\frac{d}{dt}\left\{D^3\int_\Omega m|u|^2+H^2+\frac{(P-\bar P)^2}{\bar B_2}+m|u_t|^2+H_t^2+\frac{(P_t-\bar P_t)^2}{\bar B_2}dx\right.\\ &+D\int_\Omega m|\nabla u\chi_0|^2+\frac{|\nabla P\chi_0|^2}{\bar B_2}+|\nabla H\chi_0|^2dx+D\int_\Omega\left|\nabla^2 u\chi_0\right|^2+\left|\nabla^2 H\chi_0\right|^2dx\\ &\left.+\sum_{j=1}^N Dm|\partial u\chi_j|^2+\frac{|\partial P\chi_j|^2}{\bar B_2}+|P_{y_3}\chi_j|^2dy\right\}+D^2\left(\|\nabla u\|_{L^2}^2+\|\nabla H\|_{L^2}^2+\|\nabla u_t\|_{L^2}^2+\|\nabla H_t\|_{L^2}^2\right)\\ &+\sum_{j=1}^N D|\partial\nabla u\chi_j|^2+\left|\partial\frac{dP}{dt}\chi_j\right|^2+\left|\left(\frac{dP}{dt}\right)_{y_3}\chi_j\right|^2+\left\|\nabla\frac{dP}{dt}\right\|_{L^2}dy\\ \leq & CD\varepsilon\left(\left\|\nabla^2 u\right\|_{L^2}^2+\left\|\nabla^2 P\right\|_{L^2}^2+\left\|\nabla^2 H\right\|_{L^2}^2\right)+\frac{C}{D}\left(\left\|\nabla^2 u\right\|_{L^2}^2+\|\nabla P\|_{L^2}^2\right)+CD^3\varepsilon\left\|\nabla^2 u\right\|_{L^2}^2.\end{aligned} \tag{3.52}$$

Substituting equation (3.46) into the above equation and using $\frac{dP}{dt} = -B_2 \text{div} u$, Young's inequality, poincaré's inequality, we can find

$$
\begin{aligned}
&\frac{d}{dt}\left\{ D\int_{\Omega} m\left|u\right|^2 + H^2 + \frac{(P-\bar{P})^2}{\bar{B}_2} + m\left|u_t\right|^2 + H_t^2 + \frac{(P_t-\bar{P}_t)^2}{\bar{B}_2} dx \right.\\
&+ D\int_{\Omega} m\left|\nabla u\chi_0\right|^2 + \frac{\left|\nabla P\chi_0\right|^2}{\bar{B}_2} + \left|\nabla H\chi_0\right|^2 dx + \int_{\Omega}\left|\nabla^2 u\chi_0\right|^2 + \left|\nabla^2 H\chi_0\right|^2 dx\\
&\left. + \sum_{j=1}^{N}\int_{\Omega_j^{-1}} m\left|\partial u\chi_j\right|^2 + \frac{\left|\partial P\chi_j\right|^2}{\bar{B}_2} + \left|P_{y_3}\chi_j\right|^2 dy \right\} + D\left(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2\right)\\
&+ \sum_{j=1}^{N}\int_{\Omega_j^{-1}}\left|\partial\nabla u\chi_j\right|^2 + \left\|\nabla^2 u\right\|_{L^2}^2 + \left\|\nabla^2 H\right\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 + \left\|\nabla\frac{dP}{dt}\right\|_{L^2}^2 dy\\
\leq& CD\varepsilon\left\|\nabla^2 P\right\|_{L^2}^2 + C\left\|\frac{dP}{dt}\right\|_{L^2}^2\\
\leq& CD\varepsilon\left\|\nabla^2 P\right\|_{L^2}^2,
\end{aligned}
\tag{3.53}
$$

where D is enough large, ε is arbitrarily small.

Step2: In this step, we will estimate the higher order derivatives for (P, u, H). Let $k = 0, l = 1$ in (3.37), by $D \times ((3.26) + (3.35)) + (3.37)$, we get

$$
\begin{aligned}
&\frac{d}{dt}\left\{\int_{\Omega} m\left|\nabla^2 u\chi_0\right| + \frac{\left|\nabla^2 P\chi_0\right|^2}{\bar{B}_2} + \left|\nabla^2 H\chi_0\right|^2 dx\right.\\
&\left. + \sum_{j=1}^{N}\int_{\Omega_j^{-1}} Dm\left|\partial^2 u\chi_j\right|^2 + \frac{\left|\partial^2 P\chi_j\right|^2}{\bar{B}_2} + \left|\partial P_{y_3}\chi_j\right|^2 dy\right\}\\
&+ D\int_{\Omega}\left|\nabla^3 u\chi_0\right|^2 + \left|\nabla^3 H\chi_0\right|^2 dx + \int_{\Omega}\left|\nabla^2\frac{dP}{dt}\chi_0\right|^2 dx\\
&+ \sum_{j=1}^{N}\int_{\Omega_j^{-1}}\left|\partial^2\nabla u\chi_j\right|^2 + \left|\partial\nabla\frac{dP}{dt}\chi_j\right|^2 dy\\
\leq& CD\varepsilon\left(\|\nabla u\|_2^2 + \|\nabla H\|_2^2 + \|\nabla P\|_1^2 + \|\nabla u_t\|_{L^2}^2\right)\\
&+ CD\left\|\nabla^2 u\right\|_{L^2}\left(\|\nabla u\|_2 + \left\|\nabla^2 P\right\|_{L^2}\right) + CD\left(\left\|\nabla^2 u\right\|_{L^2}^2 + \left\|\nabla^2 H\right\|_{L^2}^2\right)\\
&+ C\left(\|\nabla u\|_1^2 + \|u_t\|_1^2\right)
\end{aligned}
\tag{3.54}
$$

Let $k = 1, l = 0$ in (3.37) and together with (3.3) implies that

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega_j^{-1}}\left|\partial_{y_3}^2 P\chi_j\right|^2 dy + \int_{\Omega_j^{-1}}\left|\partial_{y_3}^2\left(\frac{dP}{dt}\right)\chi_j\right|^2 dy\\
\leq& C\left(\|\nabla u\|_1^2 + \|u_t\|_1^2 + \|\nabla P\|_{L^2}^2\right) + C\varepsilon\left(\|\nabla H\|_1^2 + \left\|\nabla^3 u\right\|_{L^2}^2\right) + C\int_{\Omega_j^{-1}}\left|\partial\nabla\frac{dP}{dt}\chi_j\right|^2 dy
\end{aligned}
\tag{3.55}
$$

By the same way, $D \times (3.54) + (3.55)$, we have

$$\begin{aligned}
&\frac{d}{dt}\left\{D^2\int_{\Omega} m\left|\nabla^2 u\chi_0\right| + \left|\nabla^2 H\chi_0\right|^2 dx + \sum_{j=1}^{N}\int_{\Omega_j^{-1}} D^2 m\left|\partial^2 u\chi_j\right|^2 + \frac{\left|\nabla^2 P\chi_0\right|^2}{\bar{B}_2}dy\right\}\\
&+D^2\int_{\Omega}\left|\nabla^3 u\chi_0\right|^2 + \left|\nabla^3 H\chi_0\right|^2 dx + \int_{\Omega}\left|\nabla^2\frac{dP}{dt}\chi_0\right|^2 dx\\
&+\sum_{j=1}^{N}\int_{\Omega_j^{-1}} D\left|\partial^2\nabla u\chi_j\right|^2 + \left|\nabla^2\frac{dP}{dt}\chi_j\right|^2 dy \qquad (3.56)\\
\leq& CD\varepsilon\left(\|\nabla u\|_2^2 + \|\nabla H\|_2^2 + \|\nabla P\|_1^2 + \|\nabla u_t\|_{L^2}^2\right)\\
&+CD^2\left\|\nabla^2 u\right\|_{L^2}\left(\|\nabla u\|_2 + \left\|\nabla^2 P\right\|_{L^2}\right) + CD^2\left(\left\|\nabla^2 u\right\|_{L^2}^2 + \left\|\nabla^2 H\right\|_{L^2}^2\right)\\
&+CD\left(\|\nabla u\|_1^2 + \|u_t\|_1^2\right) + C\|\nabla P\|_{L^2}^2.
\end{aligned}$$

Applying (2.2) to (3.48)

$$\begin{aligned}
&\left(\left\|\nabla^3 u\right\|_{L^2}^2 + \left\|\nabla^3 H\right\|_{L^2}^2\right) + \left\|\nabla^2 P\right\|_{L^2}^2\\
\leq& C\varepsilon\left\|\nabla^2 H\right\|_{L^2}^2 + \left(\left\|\nabla\frac{dP}{dt}\right\|_1^2 + \|u_t\|_1^2 + \|\nabla u\|_1^2\right. \qquad (3.57)\\
&+\left.\left\|\nabla^3 u\right\|_{L^2}^2\|\nabla u\|_1^2 + \|\nabla H_t\|_{L^2}^2 + \|\nabla P\|_{L^2}^2\right)
\end{aligned}$$

Step3: Establish the energy inequality of Gronwall-type. Consider $D^3 \times (3.53) + D \times (3.56) + (3.57)$, by poincar*é*'s inequality, there exist a function

$$Q(P,u,H) = \left\|P - \bar{P}\right\|_2^2 + \|u\|_2^2 + \|H\|_2^2 + \left\|P_t - \bar{P}_t\right\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \|H_t\|_{L^2}^2$$

such that for any positive constant C_1, we have

$$\frac{dQ}{dt} + C_1 Q + \left(\left\|\nabla^3 u\right\|_{L^2}^2 + \left\|\nabla^3 H\right\|_{L^2}^2\right) \leq 0, \qquad (3.58)$$

Integrating the above inequality over [0,t] get (1.9). By Gronwall's inequality, yields

$$Q(P,u,H) \leq C_2 Q(P_0 - \bar{P}_0, u_0, H_0)e^{-C_1 t}, \qquad (3.59)$$

which together with $(1.10)_2$ yield (1.8).

Step4: Finally, we prove the estimate of c(t). By symmetry and some tedious but straightforward calculation, we can concludes the energy estimates on c as following:

$$\frac{d}{dt}\|c\|_2^2 \leq C\|u\|_2\|c\|_2^2 + \|B_1\|_2^2,$$

By Gronwall's inequality, we get

$$\|c\|_2^2 \leq C_1\exp\left\{C_1\int_0^t\|u(\tau)\|_2 d\tau\right\}\left(\|u_0\|_2 + \int_0^t\|B_1\|_2^2 d\tau\right),$$

Using (1.8) and (1.9),we have

$$\|c\|_2 \leq C_1\exp\left\{C_1\left\|(P_0 - \bar{P}_0, u_0, H_0)\right\|_2\right\}\left(\|u_0\|_2 + \left\|(P_0 - \bar{P}_0, u_0, H_0)\right\|_2^2\right)$$

This proved the (1.17). In combination with the above, we have finished the proof of Theorem 1.1.

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