Field Approach for General Relativity

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Abstract. The general theory of relativity is based on expressing gravity by means of the metric tensor and its elements instead of some fields as in electrodynamics. This work starts with by defining some vector fields for metric or metric tensor. After metric and metric tensor are expressed in terms of these fields, the geodesic equations and Einstein equations are derived for these fields. Finally, perihelion precession and light deflection are recalculated, as two different applications of the introduced fields.

Keywords: metric tensor, geodesic equations, Lorentz force, Einstein equations, Maxwell equations Schwarzschild metric, perihelion precession, light deflection.

1 Introduction

The general theory of relativity has been one of the most important theories of the last century. The success of this theory in explaining some events drew the attention of scientists to this theory, so it gained great importance in physics. In fact, the most significant reason for this is that it brings a different understanding to gravity compared to other interactions. According to general relativity, gravity is due to the curvature in space-time, and in order to explain this interaction properly, it is necessary to calculate the curvature that extends to Einstein's equations. Another insight brought by general relativity was that gravity cannot be expressed in terms of simple fields such as electromagnetism, and be best investigated with a metric tensor and its elements.

Despite all its success and originality, general relativity has some unsolved difficulties. unification with other interactions, nonlinear equations and differential equations, role and properties of elementary particles in general relativity are some of them. However, understanding of the theory and having to perform quite complex and lengthy calculations to reach a solution on any subject can be the most difficult part. Efforts to find solutions to these challenges have been ongoing for almost a century [1].

In this paper, we present a new approach to the general theory of relativity by introducing some vector fields with the intention of finding an option to the mentioned challenges. Firstly, we will introduce some vector fields to the space-time metric. After giving some properties of these fields, we will find the geodesic equations for these fields. Next, we will calculate the Einstein equations for these fields. The fact that the geodesic equations and Einstein's equations for these fields are very interestingly related to electrodynamics, will lead us to the conclusion that these fields must satisfy Maxwell's equations. As a result of this fact, assuming a density-mass flux density similar to the charge-current density in electrodynamics, performing some applications for the introduced fields will be the last step of our study. After deriving an alternative metric to the Schwarzschild metric using the fields in question, we will try to demonstrate the consistency of our approach with two known applications: perihelion motion of a planet and deflection of light.

2 Introducing Fields

The metric of the space time is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu , \tag{1}$$

where $g_{\mu\nu}$ is the metric tensor of the space-time and its elements are some functions of the space-time and $\mu, \nu = 0,1,2,3$. In fact, (1) can be considered as a result of the inner product of differential of four-position vector as

$$ds^2 = d\mathbf{R} \bullet d\mathbf{R} ,$$

where $\mathbf{R}$ is the four-position vector. Differential of $\mathbf{R}$ can be written as

$$d\mathbf{R} = \nabla_{\mu}\mathbf{R}dx^{\mu} ,$$

where ∇_{μ} covariant derivative operator. By letting $\mathbf{A}_{\mu} = \nabla_{\mu}\mathbf{R}$, the metric becomes

$$ds^2 = \mathbf{A}_{\mu}\bullet\mathbf{A}_{\nu}dx^{\mu}dx^{\nu} .$$

Thus, the metric tensor is

$$g_{\mu\nu} = \mathbf{A}_{\mu}\bullet\mathbf{A}_{\nu} . \quad (2)$$

Naturally, an inverse metric tensor can be defined which satisfy

$$\delta_{\alpha}^{\beta} = g_{\alpha\lambda}g^{\lambda\beta} , \quad (3)$$

where δ_{α}^{β} is the Kronecker delta. Similar to (2) we can define an inverse metric tensor as

$$g^{\mu\nu} = \mathbf{A}^{\mu}\bullet\mathbf{A}^{\nu} , \quad (4)$$

where $\mathbf{A}^{\mu}$ are vectors of $g^{\mu\nu}$, or inverse vectors of $\mathbf{A}_{\mu}$.

We have introduced some vector fields to metric tensor with (2). The main subject in this study will be these fields and the results we will obtain from them. However, before starting the main subject, it would be helpful to give a few mathematical identities and properties about these fields.

Firstly, from (2), (3) and (4), we can write

$$\mathbf{A}_{\alpha}\bullet\mathbf{A}^{\beta} = \delta_{\alpha}^{\beta} . \quad (5)$$

Secondly, using (5)

$$\partial_{\mu}\left(\mathbf{A}_{\alpha}\bullet\mathbf{A}^{\beta}\right) = 0 ,$$

$$\partial_{\mu}\mathbf{A}_{\alpha}\bullet\mathbf{A}^{\beta} = -\partial_{\mu}\mathbf{A}^{\beta}\bullet\mathbf{A}_{\alpha} \quad (6)$$

can be found.

Thirdly, we derive a new equation for vector fields by means of simple mathematical calculations. Let us define a quantity as $V_{\mu}^{\ \beta} = \mathbf{V}_{\mu}\bullet\mathbf{A}^{\beta}$, where $\mathbf{V}_{\mu}$ is any vector field. Since $\mathbf{A}^{\beta} = g^{\alpha\beta}\mathbf{A}_{\alpha}$ using (4)

$$V_{\mu}^{\ \beta} = \mathbf{V}_{\mu}\bullet\mathbf{A}^{\beta} = \mathbf{V}_{\mu}\bullet\mathbf{A}_{\alpha}\left(\mathbf{A}^{\alpha}\bullet\mathbf{A}^{\beta}\right),$$

$$\mathbf{V}_{\mu} = \left(\mathbf{V}_{\mu}\bullet\mathbf{A}_{\alpha}\right)\mathbf{A}^{\alpha}$$

can be written. Also, we can write

$$\mathbf{V}_{\mu} = \left(\mathbf{V}_{\mu}\bullet\mathbf{A}^{\alpha}\right)\mathbf{A}_{\alpha} .$$

For the next identity let $V = V_{\mu}^{\ \mu} = \mathbf{V}_{\mu}\bullet\mathbf{A}^{\mu}$. Then $\frac{\mathbf{V}_{\mu}\bullet\mathbf{A}^{\mu}}{V} = 1$. As a result of (5) $\mathbf{A}_{\mu}\bullet\mathbf{A}^{\mu} = 4$ we can write

$$\frac{\mathbf{V}_{\mu}}{V} = \frac{1}{4}\mathbf{A}_{\mu} ,$$

$$\mathbf{V}_{\mu} = \frac{V}{4}\mathbf{A}_{\mu} ,$$

$$\mathbf{V}_{\mu}\bullet\mathbf{A}_{\nu} = \frac{V}{4}\mathbf{A}_{\mu}\bullet\mathbf{A}_{\nu} ,$$

$$\mathbf{V}_{\mu}\bullet\mathbf{A}_{\nu} = \frac{V}{4}g_{\mu\nu} . \quad (7)$$

The last equation can be obtained by starting from

$$\partial_{\mu}\mathbf{A}_{\nu}\bullet\mathbf{A}^{\alpha} = g_{\mu\lambda}\partial^{\lambda}\mathbf{A}_{\nu}\bullet\mathbf{A}^{\alpha} ,$$

$$\partial_\mu \mathbf{A}_\nu \bullet \mathbf{A}^\alpha = \mathbf{A}_\mu \bullet \left(\mathbf{A}_\lambda \partial^\lambda \mathbf{A}_\nu \bullet \mathbf{A}^\alpha \right),$$

$$\partial_\mu \mathbf{A}_\nu \bullet \mathbf{A}^\alpha = \mathbf{A}_\mu \bullet \partial^\lambda \mathbf{A}_\nu \left(\mathbf{A}_\lambda \bullet \mathbf{A}^\alpha \right).$$

Using (5)

$$\partial_\mu \mathbf{A}_\nu \bullet \mathbf{A}^\alpha = \partial^\alpha \mathbf{A}_\nu \bullet \mathbf{A}_\mu . \tag{8}$$

Similarly, we can write

$$\partial_\mu \mathbf{A}_\nu \bullet \mathbf{A}_\alpha = \partial_\alpha \mathbf{A}_\nu \bullet \mathbf{A}_\mu .$$

3 Geodesic Equations

We can rewrite (1) as

$$\frac{ds^2}{d\tau^2} = g_{\mu\nu} \frac{dx^\mu}{d\tau} \cdot \frac{dx^\nu}{d\tau},$$

where τ is the proper time. $\frac{ds^2}{d\tau^2}$ can be chosen as any constant for geodesic motion, say a^2. Thus

$$\frac{ds^2}{d\tau^2} = g_{\mu\nu} \frac{dx^\mu}{d\tau} \cdot \frac{dx^\nu}{d\tau} = a^2 .$$

Let $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$. Hence

$$g_{\mu\nu} \dot{x}^\mu \cdot \dot{x}^\nu = a^2 .$$

As a result of $g_{\mu\nu} g^{\mu\nu} = 4$, we can deduce that

$$\dot{x}^\mu \cdot \dot{x}^\nu = \frac{a^2}{4} g^{\mu\nu}, \tag{9}$$

$$\dot{x}^\mu \mathbf{n}^{(\mu)} = \frac{a}{2} \mathbf{A}^\mu, \tag{10}$$

where $\mathbf{n}^{(\mu)}$ is the unit vector along $\mathbf{A}^\mu$ (Parentheses mean no addition over the repeated indices).

The well-known geodesic equation in terms of metric tensor is [2]

$$\frac{d^2 x^\mu}{d\tau^2} = -\frac{1}{2} g^{\mu\lambda} \left(\partial_\beta g_{\lambda\alpha} + \partial_\alpha g_{\lambda\beta} - \partial_\lambda g_{\alpha\beta} \right) \dot{x}^\alpha \cdot \dot{x}^\beta .$$

By writing the metric and inverse metric in terms of fields, we get

$$\frac{d^2 x^\mu}{d\tau^2} = -\frac{1}{2} g^{\mu\lambda} \left(\partial_\beta \mathbf{A}_\lambda \bullet \mathbf{A}_\alpha + \mathbf{A}_\lambda \bullet \partial_\beta \mathbf{A}_\alpha + \partial_\alpha \mathbf{A}_\lambda \bullet \mathbf{A}_\beta + \mathbf{A}_\lambda \bullet \partial_\alpha \mathbf{A}_\beta - \partial_\lambda \mathbf{A}_\alpha \bullet \mathbf{A}_\beta - \mathbf{A}_\alpha \bullet \partial_\lambda \mathbf{A}_\beta \right) \dot{x}^\alpha \cdot \dot{x}^\beta .$$

Using (8) we obtain

$$\frac{d^2 x^\mu}{d\tau^2} = -\frac{1}{2} g^{\mu\lambda} \left(\partial_\beta \mathbf{A}_\lambda \bullet \mathbf{A}_\alpha + \partial_\alpha \mathbf{A}_\lambda \bullet \mathbf{A}_\beta \right) \dot{x}^\alpha \cdot \dot{x}^\beta .$$

Using (9) we can rewrite the geodesic equation as

$$\frac{d^2 x^\mu}{d\tau^2} = -\frac{a^2}{8} g^{\mu\lambda} \left(\partial_\alpha \mathbf{A}_\lambda \bullet \mathbf{A}^\alpha + \partial_\alpha \mathbf{A}_\lambda \bullet \mathbf{A}^\alpha \right) \tag{11}$$

or

$$\frac{d^2 x^\mu}{d\tau^2} = -\frac{a^2}{4} g^{\mu\lambda} \partial_\alpha \mathbf{A}_\lambda \bullet \mathbf{A}^\alpha . \tag{12}$$

Although (11) and (12) are the same, we will write (11) in a different form to find the results we are more familiar with. Obviously, this will not change the physical meaning of the result, but it will make the result more understandable. Using (6) and (8), (11) turns into

$$\frac{d^2 x^\mu}{d\tau^2} = \frac{a^2}{8} \left(\partial^\alpha \mathbf{A}_\alpha \bullet \mathbf{A}^\mu - \partial^\alpha \mathbf{A}^\mu \bullet \mathbf{A}_\alpha \right),$$

$$\frac{d^2x^\mu}{d\tau^2} = -\frac{a^2}{8}\left(-\partial^\alpha \mathbf{A}_\alpha \bullet \mathbf{A}^\mu - \partial^\alpha \mathbf{A}_\alpha \bullet \mathbf{A}^\mu\right).$$

Let $\mathbf{F}^\mu{}_\alpha = \partial^\mu \mathbf{A}_\alpha - \partial_\alpha \mathbf{A}^\mu$. Thus

$$\frac{d^2x^\mu}{d\tau^2} = \frac{a^2}{8}\mathbf{F}^\mu{}_\alpha \bullet \mathbf{A}^\alpha .$$

And using (10)

$$\frac{d^2x^\mu}{d\tau^2} = \frac{a}{4}\mathbf{F}^\mu{}_\alpha \bullet \left(\dot{x}^\alpha \mathbf{n}^{(\alpha)}\right),$$

$$\frac{d^2x^\mu}{d\tau^2} = \frac{a}{4}F^\mu{}_\alpha \dot{x}^\alpha , \tag{13}$$

where $F^\mu{}_\alpha = \mathbf{F}^\mu{}_\alpha \bullet \mathbf{n}^{(\alpha)}$. It is remarkable that (13) is the same as the covariant form of the Lorentz force [3].

4 Einstein Equations

The Christoffel symbols for the metric tensor given by (2) are

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(\partial_\mu \mathbf{A}_\nu \bullet \mathbf{A}_\beta + \partial_\mu \mathbf{A}_\beta \bullet \mathbf{A}_\nu + \partial_\nu \mathbf{A}_\mu \bullet \mathbf{A}_\beta + \partial_\nu \mathbf{A}_\beta \bullet \mathbf{A}_\mu - \partial_\beta \mathbf{A}_\mu \bullet \mathbf{A}_\nu - \partial_\beta \mathbf{A}_\nu \bullet \mathbf{A}_\mu\right).$$

Using identities derived for vector fields in the Chapter 2, Christoffel symbols can be written in some different forms. These are

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2}\left(\partial_\nu \mathbf{A}_\mu + \partial_\mu \mathbf{A}_\nu\right)\bullet \mathbf{A}^\alpha ,$$

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2}\mathbf{F}^\alpha{}_\mu \bullet \mathbf{A}_\nu ,$$

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2}\mathbf{F}^\alpha{}_\nu \bullet \mathbf{A}_\mu .$$

After some calculations, and using the above Christoffel symbols and $\partial_\beta \mathbf{F}^\alpha{}_\nu + \partial_\nu \mathbf{F}_\beta{}^\alpha + \partial^\alpha \mathbf{F}_{\nu\beta} = 0$ the Rienman tensor of the fields can be found as

$$R^\alpha{}_{\mu\beta\nu} = \frac{1}{2}\partial^\alpha \mathbf{F}_{\beta\nu} \bullet \mathbf{A}_\mu + \frac{1}{4}\mathbf{F}^\alpha{}_\nu \bullet \mathbf{F}_{\beta\mu} + \frac{1}{4}\mathbf{F}^\alpha{}_\beta \bullet \mathbf{F}_{\mu\nu} .$$

Similarly, the Ricci Tensor is

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} = \frac{1}{2}\partial^\alpha \mathbf{F}_{\alpha\nu} \bullet \mathbf{A}_\mu + \frac{1}{4}\mathbf{F}^\alpha{}_\nu \bullet \mathbf{F}_{\alpha\mu} + \frac{1}{4}\mathbf{F}^\alpha{}_\alpha \bullet \mathbf{F}_{\mu\nu} .$$

Let $\mathbf{J}_\nu = \frac{1}{2}\partial^\alpha \mathbf{F}_{\alpha\nu} = \partial^\alpha \partial_\alpha \mathbf{A}_\nu$, which is nothing but Maxwell equations. The Ricci scalar is

$$R = R_{\alpha\beta}g^{\alpha\beta} = \mathbf{J}_\beta \bullet \mathbf{A}^\beta + \frac{1}{8}\mathbf{F}^{\alpha\beta} \bullet \mathbf{F}_{\alpha\beta} .$$

In the last equation $\mathbf{F}^{\alpha\beta} \bullet \mathbf{F}_{\alpha\beta}$ is multiplied by $\frac{1}{2}$ due to twofold summation. Thus, Einstein tensor can be written as

$$G_{\mu\nu} = \mathbf{J}_\nu \bullet \mathbf{A}_\mu + \frac{1}{4}\mathbf{F}^\alpha{}_\nu \bullet \mathbf{F}_{\alpha\mu} - \frac{1}{2}g_{\mu\nu}\left(\mathbf{J}_\alpha \bullet \mathbf{A}^\alpha + \frac{1}{8}\mathbf{F}^{\alpha\beta} \bullet \mathbf{F}_{\alpha\beta}\right).$$

From Equation (7) we can write,

$$\mathbf{J}_\mu \bullet \mathbf{A}_\nu = \frac{1}{4}\left(\mathbf{J}_\beta \bullet \mathbf{A}^\beta\right)g_{\mu\nu} .$$

By putting the last equation in Einstein tensor, we have

$$G_{\mu\nu} = \frac{1}{4}\left[\mathbf{F}^{\alpha}{}_{\nu}\bullet\mathbf{F}_{\alpha\mu} - g_{\mu\nu}\left(\frac{1}{4}\mathbf{F}^{\alpha\beta}\bullet\mathbf{F}_{\alpha\beta} + \mathbf{J}_{\alpha}\bullet\mathbf{A}^{\alpha}\right)\right],$$

which is nothing but the stress-energy tensor of electrodynamics in presence of charge-current density [4].

5 Applications

In the general theory of relativity, to obtain any practical result of a system it is necessary to perform some repetitious calculations for the system under study. Additionally, physical explanations, equations and solutions are required for at least 10 elements of the Einstein tensor. However, this is a very complex, long and demanding process that every physicist is well aware of.

We have seen that the vector fields we introduced to the metric tensor in previous chapters, provide Maxwell's equations such as scalar and vector potentials in electromagnetism. This fact leads us to determine these vector fields using $\mathbf{J}_{\nu}$, as in electrodynamics, and then determine the metric tensor using these vector fields. So, we will focus on $\mathbf{J}_{\nu}$ as the basic element of any interaction. Although it is known as charge-current density in electrodynamics, we will accept it to be mass-mass flux density for gravitation. By this way we will try to achieve the desired result without solving Einstein's equations or any other complex equations. Thus, we hope to overcome the difficulties of the general theory of relativity that we have mentioned above.

Before we start applications, it should be noted that to keep this study short, we will not do calculations for gravitational time dilation and redshift calculations, which are simpler than the two examples we will do. One can clearly see that our metric will give the same result with Schwarzschild metric results. We can now begin to apply our approach to the two special events of general relativity we know best.

For the both applications, we are considering a point outside of a constant mass planet or star. Additionally, we assume that since there is no mass change or flux for the sun, the spatial vectors must provide flat space. So, for these physical conditions we have to set $\mathbf{J}_{\nu} = 0$ and $\mathbf{A}_1 = \mathbf{A}_2 = \mathbf{A}_3 = \mathbf{1}$. Thus, all we need to determine is $\mathbf{A}_0$ which is static and depends only on r. Then the equation to be solved is

$$\partial^{\mu}\partial_{\mu}\mathbf{A}_0 = \mathbf{0}\,.$$

Solution of it in spherical coordinates is

$$\mathbf{A}_0(r) = \left(D - \frac{R}{r}\right)\mathbf{n}_0\,,$$

where D and R integration constants. Then we can write a spherical symmetric metric as

$$ds^2 = -c^2\left(D - \frac{R}{r}\right)^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 Sin^2\theta d\varphi^2\,.$$

We can determine D and R easily. To get flat space-time when $r \to \infty$, we find $D = 1$. And we can use Newtonian limit to determine R. In order to get classical mechanics lagrangian for small mass, $r \to \infty$ and $v \ll c$, it should be as

$$R = \frac{GM}{c^2}\,,$$

where G is universal gravitational constant, M is the mass of sun or planet and c is speed of light. Note that R is half of the Schwarzschild radius [5]. Accordingly, the metric will be

$$ds^2 = -c^2\left(1 - \frac{R}{r}\right)^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 Sin^2\theta d\varphi^2\,. \qquad (14)$$

The metric given by (14) is the metric that our approach gives as an alternative to the Schwarzschild metric [6], and it will be main tool in our applications.

5.1 Perihelion Precession of a Planet

In order to calculate perihelion precession of a planet in the solar system, geodesic equations of (14) must be found and solved. Two solutions that will be useful to us as a result of the known processes, are respectively;

$$\dot{t} = \varepsilon\left(1 - \frac{R}{r}\right)^{-2}, \tag{15}$$

$$\dot{\varphi} = \frac{j}{r^2}. \tag{16}$$

where dots denote $\frac{d}{d\tau}$, and ε, j are integration constants. We continue with deriving a simpler geodesic equation for r. For this, we divide (14) by $d\tau^2$, and choose $\left(\frac{ds}{d\tau}\right)^2 = -1$. Consequently, we have

$$-1 = -c^2\left(1 - \frac{R}{r}\right)^2 \dot{t}^2 + \dot{r}^2 + r^2\dot{\theta}^2 + r^2 Sin^2\theta\dot{\varphi}^2.$$

For a planet in the solar system, we can choose axes such that $\theta = \frac{\pi}{2}$, which allows us to write $\dot{\theta} = 0$. Using this and placing (15) and (16) into the last equation we get

$$-1 = -c^2\varepsilon^2\left(1 - \frac{R}{r}\right)^{-2} + \dot{r}^2 + \frac{j^2}{r^2}.$$

In order to get a simpler equation, we change $u = \frac{1}{r}$ and we write derivates with respect to φ. After the required calculations, the last equation becomes

$$-1 = -c^2\varepsilon^2\left(1 - Ru\right)^{-2} + j^2u'^2 + j^2u^2, \tag{17}$$

where the prime denotes $\frac{d}{d\varphi}$. The integration constant ε can be determined by using (17). Since (17) is valid for all r, when $r \to \infty$ $u = \frac{1}{r} \to 0$ and $u' = -\frac{r'}{r^2} \to 0$. Thus

$$\varepsilon^2 = 1/c^2.$$

So, we can rewrite (17) as

$$-1 = -\left(1 - Ru\right)^{-2} + j^2u'^2 + j^2u^2. \tag{18}$$

Differentiation of (18) with respect to φ yields

$$u'' + u = \frac{R}{j^2}\left(1 - Ru\right)^{-3},$$

which is the equation we try to find. Inappropriately, the last equation is a non-linear differential equation and cannot be solved exactly. But fortunately, since $Ru \ll 1$ for the solar system we can solve it by doing some approximations. For this, first of all, we expand $\left(1 - Ru\right)^{-3}$ as $\left(1 - Ru\right)^{-3} \approx 1 + 3Ru - 3R^2u^2$. Then it turns into

$$u'' + \left(1 - \frac{3R^2}{j^2}\right)u = \frac{R}{j^2} - \frac{3R^3}{j^2}u^2. \tag{19}$$

The reason we do not neglect the term $\frac{3R^3u^2}{j^2}$ is to compare the calculations with the same order results obtained with the Schwarzschild metric. In order to see what result we will find, we will first solve (19) neglecting $\frac{3R^3u^2}{j^2}$, since it is too small for any planet in the solar system. Thus, (19) becomes

$$u'' + \left(1 - \frac{3R^2}{j^2}\right)u = \frac{R}{j^2}. \tag{20}$$

Since we investigate the path of a planet in the solar system, we can deduce that j^2 must be such that:

$$\frac{R}{j^2} = \frac{1}{a(1-e^2)},$$

$$\frac{R^2}{j^2} = \frac{R}{a(1-e^2)},$$

where a is the orbit major semi axis, and e is the eccentricity of the planet. (20) can be solved easily and u can be found as

$$u \cong \frac{1 - eCos\tilde{\varphi}}{a(1-e^2)},$$

$$\tilde{\varphi} = \sqrt{1 - \frac{3R}{a(1-e^2)}}\varphi.$$

If $\tilde{\varphi}$ is compared to the results of classical mechanics, we can see that it already includes the perihelion motion. Let us calculate its amount now. Since $\frac{3R}{a(1-e^2)} \ll 1$ for the solar system we can write

$$\sqrt{1 - \frac{3R}{a(1-e^2)}} \approx 1 - \frac{1}{2}\frac{3R}{a(1-e^2)},$$

$$\tilde{\varphi} \cong \varphi\left(1 - \frac{3GM}{2ac^2(1-e^2)}\right),$$

The perihelion precession can be found by using the last equation as (and using that for complete orbit $\varphi = 2\pi$)

$$\delta^* \cong \frac{3\pi GM}{ac^2(1-e^2)},$$

which is inconsistent with observations and prediction of general relativity. This fact shows us that we need to solve (19) as it is without neglecting any term. In this case it turns into a well-known perturbation problem [9] and if we solve it using proper method, an additional amount of $\frac{3\pi GM}{ac^2(1-e^2)}$ must be added to δ^*. In this case the total perihelion precession is

$$\delta = \frac{6\pi GM}{ac^2(1-e^2)},$$

which is the same result with observations [7] and the Schwarzschild metric results [8].

5.2 Deflection of Light

The deflection of light around the sun is the result of photon deflection. Therefore, we must examine photon motions as if they were very small particles. For this we will again use the metric given in (14). However, this time the solutions of some equations and integration constants of them will naturally be different. By straightforward calculations we will get again (19), but for a photon whose natural path is a straight line, we should it write as

$$u'' + \left(1 - \frac{3R^2}{j^2}\right)u = -\frac{3R^3}{j^2}u^2 . \tag{21}$$

Since, in this application the integration constant j^2 has a different value, we must recalculate it. For this, (18) can be used again, but noting that φ runs from $-\pi/2$ to $\pi/2$ and that when $r = R_0$ we have $\varphi = 0$ and $u' = 0$ (R_0 is the smallest distance of light from the centre of the sun). If we them put into (18), what we get is

$$-1 = -\left(1 - \frac{R}{R_0}\right)^{-2} + \frac{j^2}{R_0^2} .$$

Using $R_0 \gg R$ for our solar system, j^2 is found as

$$j^2 = 2RR_0 .$$

Since the term on the right of is negligible, let's first find what the homogeneous part of (21) gives. If we substitute the value of j^2 solution of (21) is

$$u = \frac{1}{R_0} Cos \sqrt{1 - \frac{3R}{2R_0}}\varphi .$$

As a result of $R_0 \gg R$ we can write

$$u = \frac{1}{R_0} Cos\left(1 - \frac{3R}{4R_0}\right)\varphi . \tag{22}$$

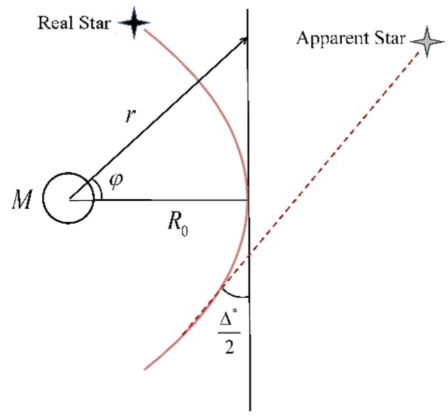

Figure 1. Trajectory of a photon according to (22) (*red curved line*) and trajectory of a free photon (*vertical straight line*).

Comparing the trajectory of a photon in (22) with that of a free photon, we see that it contains a small amount of deflection. More precisely, it means that the trajectory of the photon will not be straight line, but a curved path. The path difference between a free photon and the photon predicted by (22) is illustrated in Figure 1. In this figure, the photon trajectory resulting from (22) has been drawn in exaggeration.

Let $\bar{\varphi} = \left(1 - \frac{3R}{4R_0}\right)\varphi$. If $r \to \infty$, it is obvious that $\varphi \to \frac{\pi}{2}$ and $\bar{\varphi} \to \frac{\pi}{2} - \frac{3R}{4R_0}\frac{\pi}{2}$. The difference between $\bar{\varphi}$ and φ is nothing but $\frac{\Delta^*}{2}$. The first order deflection can be found as

$$\Delta^* = \frac{3\pi}{4}\frac{GM}{c^2 R_0} \simeq 2,35\frac{GM}{c^2 R_0} .$$

This deflection is far from observations and the results of the Schwarzschild metric, so we need to solve (21) without neglecting the term on the right of it. Again, using the proper perturbation method [9], we will obtain an additional $2\frac{GM}{c^2R_0}$ deflection. In this case total deflection will be

$$\Delta = \left(\frac{3\pi}{4}+2\right)\frac{GM}{c^2R_0} \simeq 4,35\frac{GM}{c^2R_0}.$$

This deflection is approximately 9% more than the result of the Schwarzschild metric and corresponds to about 1.90 arcsec.

6 Conclusions

While three of the four fundamental forces of physics are defined within the framework of quantum mechanics and quantum field theory, the definition of gravity with field theory has still not been fully realized. Also, Einstein's general relativity theory has been accepted as the basic and most appropriate approach for gravity. In addition, this understanding also brought about the result that gravity was almost completely separated from other interactions. So, the most important result of this study is that the metric tensor can be expressed in terms of some fields. If we recall the idea that gravity is best expressed by general relativity and the metric tensor, we can say that we have presented a better method for expressing gravity.

The vector fields we define in this study seem to be an obvious solution to the problem in question for gravity or general relativity. The metric tensor and its elements, which are accepted as a kind of field in general relativity theory, actually contain quadratic terms, which prevents general relativity from being connected to field theory. The fields we have introduced not only linearized the metric tensor, but also provided the missing fields for gravity. Using these fields, we think that gravity can be expressed more simply in field theory.

It is also interesting that while we think that only general relativity and Einstein's equations are suitable for gravity, the fields we add give the Lorentz force and the stress-energy tensor of electrodynamics. This could mean that we can also study electrodynamics with general relativity. Or we can conclude that not only gravity but also electrodynamics bends space-time. On the other hand, if we consider only gravity, these fields satisfy Maxwell's equations. In this case, these vector fields express some waves similar to fields in electrodynamics. This situation leads to the conclusion that our fields also include gravitational radiation.

Hence, we can say that gravity is not very different from other fundamental interactions, or at least it is very similar to electromagnetism. Since the fields added to the metric are added without considering any interaction, the general results found mathematically (Lorentz force and stress-energy tensor of electrodynamics) should cover for all interactions metrics. This actually introduces a new condition for metrics or interaction space-time. Elements of metric tensor must be inner products of some vector fields which satisfy Maxwell equations. We can find these vector fields according to the charge-current density or mass-mass flux density of the interaction we are examining. Since the charge-current density is a physical quantity and it contains all kinds of information about the event under investigation, we do not need think about at least 10 elements of the Einstein tensor separately and trying to link it to a physical cause. We can perform all the necessary calculations easier just using the density of the charge-current or density-mass flux density denoted by $\mathbf{J}_\nu$.

We can state many other details on the subject. However, as these will remain speculative at the moment let us express some practical results of our approach. The metric we calculated as an alternative to the Schwarzschild metric predicts gravitational time dilation, redshift and perihelion precession without any problem. However, our approach does not have the singularity of the Schwarzschild metric. In the field theory this singularity was a problem, although for the general theory of relativity this meant many new results. Also, the light deflection has a slightly different prediction than the Schwarzschild metric. While Schwarzschild metric predicts the light deflection as 1.75 arcsec, (14) predicts it as 1.90 arcsec.

The magnitude of light deflection has been a problem in general relativity for decades. Measurements made with visible light did not give the predicted result by general relativity, so measurements were made with radio waves instead of visible light. The predicted results were achieved in this way. However, the reason for the difference between the deflection of visible light and the deflection of radio waves has never been clarified. Measurement difficulties in visible light were cited as the main reason for not achieving the predicted result. When we examine the results of the measurements on the subject, we see that the results of visible light are close to our result [10] and the results of radio waves to the result of Schwarzschild metric [11][12].

References

1. S.M. Wagh, “Some fundamental issues in General Relativity and their resolution”, Available: https://arxiv.org/pdf/gr-qc/0402003.pdf
2. C. W. Misner, K. P. Thorne and J. A. Wheeler, Gravitation. Freeman and Company, 1973, pp. 216-218.
3. Covariant Formulation of Electrodynamics, Available: https://webhome.phy.duke.edu/~rgb/Class/phy319/phy319/node135.html
4. J. D. Jackson, Classical Electrodynamics. John Wiley & Sons, 1999, pp. 608-610.
5. Schwarzschild radius, Available: https://astronomy.swin.edu.au/cosmos/s/Schwarzschild+Radius
6. Schwarzschild metric, Available: https://hepweb.ucsd.edu/ph110b/110b_notes/node75.html
7. L Clemence, G. M. "The Relativity Effect in Planetary Motions". Reviews of Modern Physics, vol. 19 (4), pp. 361–364
8. C. Magnan, “Complete calculations of the perihelion precession of Mercury and the deflection of light by the Sun in General Relativity”, Available: https://arxiv.org/abs/0712.3709
9. G. Hooft, Introduction to General Relativity. Utrecht University, 1998, pp. 53-54. Available: https://webspace.science.uu.nl/~hooft101/lectures/genrel_2010.pdf
10. Texas Mauritanian Eclipse Team, “Gravitational deflection of light: solar eclipse of 30 June 1973 I. Description of procedures and final result”. Astronomical Journal, vol. 81, pp. 452-454. Available: http://articles.adsabs.harvard.edu/full/seri/AJ.../0081//0000453.000.html
11. H. C, Ohanian and R. Ruffuni, Gravitation and Spacetime. Cambridge University, 2013, pp. 138-142.
12. E. Fomalont, S. Kopeikin, G. Lanyi, J. Benson, “Progress in Measurements of the Gravitational Bending of Radio Waves Using the VLBA”, The Astrophysical Journal, Volume 699, Number 2, pp. 1395-1402. Available: https://iopscience.iop.org/article/10.1088/0004-637X/699/2/1395

