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New Generalization of Length Biased Exponential Distribution with Applications

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Abstract. In this paper, a compound continuous distribution (i.e. Exponentiated length biased exponential (E-LBE) distribution) is given. Also, the statistical properties of the proposed distribution are examined, and the parameters are obtained by maximum likelihood estimation method. The flexibility, adequacy, and superiority of the proposed model were investigated by means of applications to dataset. The result indicates that the E-LBE distribution outperformed the competing distributions considered.

Keywords: Exponential distribution, length biased, exponential generalized family of distribution, maximum likelihood estimation, hazard functions, survival function, carbon fibre dataset, aircraft windshield dataset.

1 Introduction

Lifetime processes have recently received several attentions through the modelling of their distribution. The development of more composite but flexible distributions therefore depends on how the researchers can compound one or more distributions in order to form a better or comparable distribution [1]. The exponential distribution is used widely statistically to describe the time between events in a Poisson process. This makes the exponential distribution important for a process with continuously memoryless random processes with a constant failure rate; however, in real life it is almost impossible to produce this constant failure rate. Thus, to account for this disadvantage, some exponential distribution generalizations for modeling lifetime data have been recently proposed [2-3]. In recent years many exponential distribution generalizations have been developed, such as the Marshall Olkin length biased exponential distribution [4], exponentiated exponential [5-6], generalized exponentiated moment exponential [7], extended exponentiated exponential [8], Marshall-Olkin exponential Weibull [9], Marshall-Olkin generalized exponential [3], and exponentiated moment exponential [10] distributions.

A random variable X is said to have a length biased exponential distribution with parameter β if its probability density function (pdf) and cumulative distribution function (cdf) is given by equation (1) and (2) respectively [11]:

$$g(x;\beta) = \frac{x}{\theta^2} e^{-x/\theta}, \quad x > 0, \theta > 0 \tag{1}$$

$$G(x) = 1 - \left(1 + \frac{x}{\theta}\right) e^{-x/\theta}, \quad x > 0, \theta > 0 \tag{2}$$

where θ is the scale parameter.

Cordeiro et al. [12] proposed a new way of adding two parameters to a continuous distribution. For a given continuous baseline cumulative distribution function, they defined the cdf and pdf of the exponentiated generalized family of distributions with two extra shape parameters $\alpha > 0$ and $\beta > 0$ given by

$$F(x) = \{1 - [1 - G(x)]^{\alpha}\}^{\beta} \tag{3}$$

and

$$f(x) = \alpha\beta g(x)[1 - G(x)]^{\alpha-1}\{1 - [1 - G(x)]^{\alpha}\}^{\beta-1} \tag{4}$$

respectively, where α and β are additional shape parameters.

Thus, we proposed a new generalization of the length biased exponential distribution called the Exponentiated length biased exponential (E-LBE) distribution for modeling lifetime data due to some interesting properties such as "lack of memory". In the rest of the paper, we define the E-LBE densities, the survival and hazard functions, and plots for different parameter values in Section 2; some of the statistical properties of the proposed E-LBE distribution are discussed in minute details in section 3, Application of the E-LBE distribution to lifetime data in section 4. The concluding remark is presented in section 5.

2 Exponentiated Length Biased Exponential (E-LBE) Distribution

We extend a one parameter length biased exponential distribution to a three parameter E-LBE. The density functions are clearly stated below.

The cumulative distribution function of the E-LBE distribution is given by

$$F_{E-LBE}(x) = \left\{1 - \left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha}\right\}^{\beta}, \qquad \alpha > 0, \beta > 0, \theta = 0 \tag{5}$$

The corresponding probability density function is given by

$$f_{E-LBE}(x) = \alpha\beta\theta^{-2}xe^{-x/\theta}\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha-1}\left[1-\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha}\right]^{\beta-1}, \ \alpha > 0, \beta > 0, \theta = 0 \tag{6}$$

The survival function is given by

$$S_{E-LBE}(x) = 1 - \left\{1 - \left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha}\right\}^{\beta}, \qquad \alpha > 0, \beta > 0, \theta = 0 \tag{7}$$

The hazard, reverse hazard and cumulative hazard function is given by equation (8), (9), (10) respectively

$$h_{E-LBE}(x) = \frac{\alpha\beta\theta^{-2}xe^{-x/\theta}\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha-1}\left[1-\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha}\right]^{\beta-1}}{1-\left\{1-\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha}\right\}^{\beta}}, \qquad \alpha > 0, \beta > 0, \theta = 0 \tag{8}$$

$$r_{E-LBE}(x) = \frac{\alpha\beta\theta^{-2}xe^{-x/\theta}\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha-1}\left[1-\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha}\right]^{\beta-1}}{\left\{1-\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha}\right\}^{\beta}}, \qquad \alpha > 0, \beta > 0, \theta = 0 \tag{9}$$

$$H_{E-LBE}(x) = -\log\left[1 - \left\{1 - \left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha}\right\}^{\beta}\right], \qquad \theta > 0, \gamma > 0, \beta = 0 \tag{10}$$

The odd of the E-LBE distribution is given by

$$O_{E-LBE}(x) = \frac{\left\{1-\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha}\right\}^{\beta}}{1-\left\{1-\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha}\right\}^{\beta}}, \qquad \alpha > 0, \beta > 0, \theta = 0 \tag{11}$$

The plots for different parameter values of the E-LBE distribution are given in figure 1-4 below

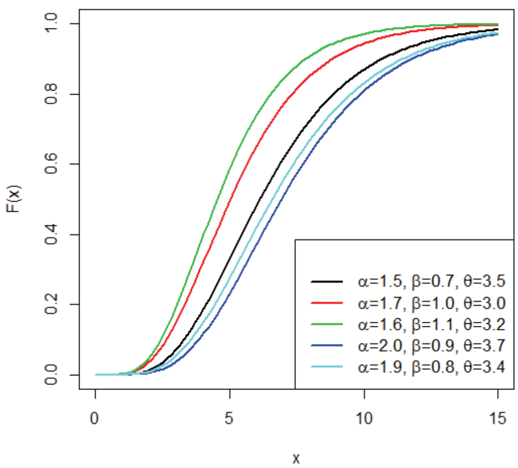

Figure 1. cdf plot of the E-LBE at different parameter values.

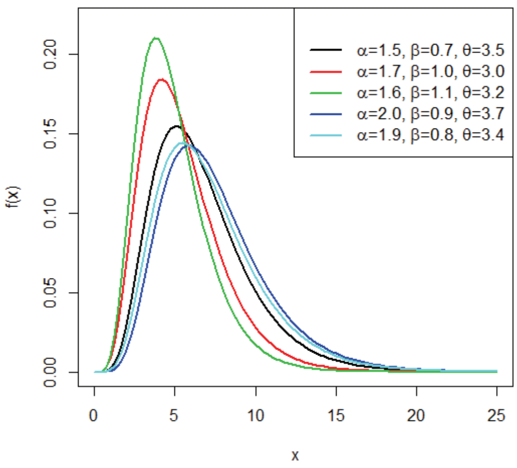

Figure 2. pdf plot of the E-LBE at different parameter values.

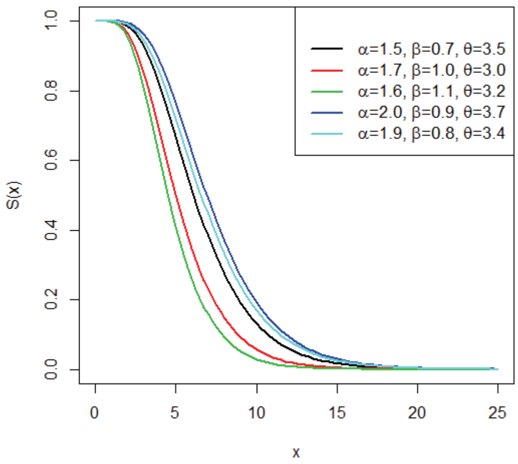

Figure 3. Survival plot of the E-LBE distribution at different parameter values.

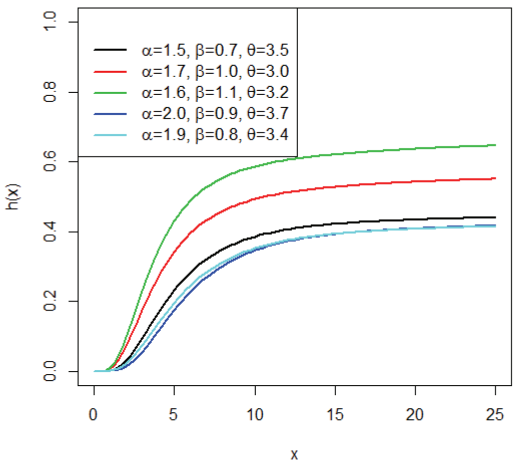

Figure 4. Hazard plot of the E-LBE distribution at different parameter values.

3 Some Statistical Properties of the E-LBE Distribution

In this section, we examine in minute details, some basic properties such as the asymptotic behavior, parameter estimation and order statistics of the E-LBE distribution.

3.1 Asymptotic Behavior

The behavior of the E-LBE model in equation (6) is examined as $x \to 0$ and as $x \to \infty$

$$\lim_{x\to 0} f_{E-LBE}(x) = \lim_{x\to 0} \alpha\beta\theta^{-2}xe^{-x/\theta}\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha-1}\left[1-\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha}\right]^{\beta-1} = 0$$

$$\lim_{x\to \infty} f_{E-LBE}(x) = \lim_{x\to \infty} \alpha\beta\theta^{-2}xe^{-x/\theta}\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha-1}\left[1-\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha}\right]^{\beta-1} = 0$$

This clearly shows that the Exponentiated length biased exponential distribution is unimodal. A clear observation of figure 2 shows the E-LBE model has only one peak. This supports our claim that the E-LBE distribution has only one mode. It can also be shown that for the E-LBE distribution, $\lim_{x\to\infty} F(x) = 1$

$$\lim_{x\to\infty} F(x) = \left\{1-\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha}\right\}^{\beta} = \left\{1-\left[(\infty)e^{-\infty/\theta}\right]^{\alpha}\right\}^{\beta} = \{1-[0]^{\alpha}\}^{\beta} = 1$$

3.2 Parameter Estimation

Using maximum likelihood estimation techniques, we estimate the unknown parameter of the E-LBE model based on a complete sample. Let $X_1, \dots X_n$ indicate a random sample of the complete E-LBE distribution data, and then the sample's likelihood function is given as

$$L = \prod_{i=1}^{n} f(x_i; \alpha, \beta, \theta)$$

$$L = \prod_{i=1}^{n} \alpha\beta\theta^{-2}x_i e^{-x_i/\theta}\left[\left(1+\frac{x_i}{\theta}\right)e^{-x_i/\theta}\right]^{\alpha-1}\left[1-\left[\left(1+\frac{x_i}{\theta}\right)e^{-x_i/\theta}\right]^{\alpha}\right]^{\beta-1}$$

The log likelihood function may be expressed as

$$L(x) = n[\ln\alpha - \ln\beta - 2\ln\theta] - \alpha\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right) + \sum_{i=1}^{n}\ln x_i + (\alpha-1)\sum_{i=1}^{n}\left(1+\frac{x_i}{\theta}\right)$$
$$+(\beta-1)\sum_{i=1}^{n}\ln\left[1-\left\{\left(1+\frac{x_i}{\theta}\right)\exp\left(\frac{-x_i}{\theta}\right)\right\}^{-\alpha}\right] \quad (12)$$

By taking the derivative with respect to $\alpha, \beta,$ and θ and fixing the outcome to zero, we have

$$\frac{\partial L(x)}{\partial\theta} = \frac{-2n}{\theta} + \alpha\sum_{i=1}^{n}\left(\frac{x_i}{\theta^2}\right) - (\alpha-1)\sum_{i=1}^{n}\left(\frac{x_i}{\theta^2}\right) + \frac{\alpha(\beta-1)\sum_{i=1}^{n}\left\{\left(1+\frac{x_i}{\theta}\right)\exp\left(\frac{-x_i}{\theta}\right)\right\}^{\alpha-1}\left\{\left(\frac{x_i}{\theta^3}\right)\exp\left(\frac{-x_i}{\theta}\right)\right\}}{1-\left\{\left(1+\frac{x_i}{\theta}\right)\exp\left(\frac{-x_i}{\theta}\right)\right\}^{-\alpha}} = 0 \quad (13)$$

$$\frac{\partial L(x)}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right) + \sum_{i=1}^{n}\left(1+\frac{x_i}{\theta}\right) - \frac{(\beta-1)\sum_{i=1}^{n}\left\{\left(1+\frac{x_i}{\theta}\right)\exp\left(\frac{-x_i}{\theta}\right)\right\}^{\alpha-1}\left\{\left(\frac{x_i}{\theta}\right)\ln\left(\frac{-x_i}{\theta}\right)\right\}}{1-\left\{\left(1+\frac{x_i}{\theta}\right)\exp\left(\frac{-x_i}{\theta}\right)\right\}^{-\alpha}} = 0 \quad (14)$$

$$\frac{\partial L(x)}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\ln\left[1-\left\{\left(1+\frac{x_i}{\theta}\right)\exp\left(\frac{-x_i}{\theta}\right)\right\}^{-\alpha}\right] = 0 \quad (15)$$

Solving equation (18)-(20) iteratively, will give the estimate of the parameters of the E-LBE model.

3.3 Order Statistics

We considered a random sample denoted by $X_1 \dots X_n$ from the densities of the E-LBE distribution. Then,

$$f_{j:n}(x) = \frac{n!}{(j-1)!\,(n-1)!} f_{E-LBE}(x)F_{E-LBE}(x)^{j-1}[1-F_{E-LBE}(x)]^{n-j}$$

The probability density function of the j^{th} order statistics for the E-LBE distribution is given as

$$f_{j:n}(x) = \frac{n!}{(j-1)!(n-j)!}\alpha\beta\theta^{-2}xe^{-x/\theta}\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha-1}\left[1-\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha}\right]^{\beta-1}$$
$$\left[\left\{1-\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha}\right\}^{\beta}\right]^{j-1}\left[1-\left\{1-\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha}\right\}^{\beta}\right]^{n-j} \quad (16)$$

The E-LBE distribution has minimum order statistics given as

$$f_{1:n}(x) = \alpha\beta\theta^{-2}xe^{-x/\theta}\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha-1}\left[1-\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha}\right]^{\beta-1}$$
$$\left[1-\left\{1-\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha}\right\}^{\beta}\right]^{n-1} \quad (17)$$

And maximum order statistics given as

$$f_{n:n}(x) = n\alpha\beta\theta^{-2}xe^{-x/\theta}\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha-1}\left[1-\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha}\right]^{\beta-1}$$
$$\left[\left\{1-\left[\left(1+\frac{x}{\theta}\right)e^{-x/\theta}\right]^{\alpha}\right\}^{\beta}\right]^{n-1} \quad (18)$$

4 Data Analysis

Here, the importance and flexibility of the Exponentiated length biased exponential distribution by comparing the results of the model fit with that of other Exponentiated- G family of distributions. Two dataset were used in this study to compare between fits of the Exponentiated length biased exponential distribution (E-LBE) with that of Exponentiated Generalized Extended Exponential (E-E) [13], Exponentiated-Lomax (E-L) [14], and, Exponentiated-Weibull (E-W) [15]. The first data set being The tensile strength of 100 carbon fibers reported by [16, 17, 18, 19], and the second being the Aircraft Windshield dataset reported by [20]. Several criterions were used to determine the distribution for the best fit: Akaike Information Criteria (AIC), Consistent Akaike Information Criteria (CAIC), Bayesian Information Criteria (BIC), and Hannan and Quinn Information Criteria (HQIC). The Shapiro Wilk (S-W), Anderson Darling (A) statistic and Kolmogorov Smirnov (KS) statistic were also provided.

Data I. Tensile strength of carbon fibre

3.7, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 0.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22, 2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65.

Table 1. Descriptive statistics

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Skewness	Kurtosis	Variance
0.39	1.830	2.675	2.601	3.197	5.560	0.3590264	3.126169	1.042162

From Table 1, we noticed that the data set is slightly positively skewed with the coefficient of Skewness being 0.3590264 and a variance of 1.042162.

Table 2. MLEs, SW, AD and K–S of parameters for carbon fibre data

Model	Estimates				SW	K-S	AD
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\gamma}$			
E-LBE (Proposed Model)	0.1258965	2.3562451	0.8021456	-	0.0512457	0.0421523	0.214563
E-E	1.000577	7.369123	1.000591	-	0.2587586	0.1159516	1.34799
E-L	7.27799443	7.73541071	0.01976084	7.27799442	0.2739396	0.1176701	1.43346
E-W	0.4738824	1.2467461	0.5004734	2.4426383	0.0821744	0.0668762	0.472650

For all competing distributions using the strength of carbon fibre data set, Table 2 shows parameter estimate and the value for the Shapiro Wilk (S-W), Anderson Darling (AD), Kolmogorov Smirnov (K-S) statistic.

Table 3. Log-likelihood, AIC, AICC, BIC and HQIC values of models fitted for carbon fibre data

Model	Negative LL	AIC	CAIC	BIC	HQIC	P-VALUE
E-LBE (Proposed Model)	141.102	287.0541	287.0711	297.4529	293.0452	0.9245884
E-E	147.027	300.0541	300.3041	307.8696	303.2172	0.1358611
E-L	147.5959	303.1957	303.6168	313.6164	307.4132	0.1253873
E-W	142.0108	292.0215	292.4426	302.4422	296.239	0.7624059

From Table 3, the E-LBE has the highest log-likelihood values of -141.102 and the lowest AIC, CAIC, BIC and HQIC values of 287.0541, 287.0711, 297.4529, and 293.0452 respectively. For this reason, it is chosen as the most appropriate model amongst the considered models.

Data II. Aircraft windshield data set

0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309,1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070,1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924, 1.281,2.038, 2.823, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432,2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506,2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619,2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757,2.324, 3.376, 4.663.

Table 4. Descriptive statistics for the aircraft windshield dataset

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Skewness	Kurtosis	Variance
0.040	1.839	2.354	2.557	3.393	4.663	0.09949403	2.347684	1.251768

From Table 4, we noticed that the data set is slightly positively skewed with the coefficient of Skewness being 0.09949403 and a variance of 1.251768.

Table 5. MLEs, SW, AD and K–S of parameters for aircraft windshield dataset

Model	Estimates				SW	K-S	AD
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\gamma}$			
E-LBE (Proposed Model)	0.1257653	4.1026541	1.4523654	-	0.0544896	0.0522182	0.021452
E-E	0.8705868	3.5604763	0.8705865	-	0.2106315	0.1209529	1.69461
E-L	7.69537742	3.62564915	0.01311515	7.69537242	0.2226253	0.1217029	1.771718
E-W	0.08926839	0.46205683	0.51865065	3.90195906	0.0812481	0.0658412	0.065841

For all competing distributions using the strength of Aircraft Windshield Dataset, Table 5 shows parameter estimate and the value for the Shapiro Wilk (S-W), Anderson Darling (AD), Kolmogorov Smirnov (K-S) statistic.

Table 6. Log-likelihood, AIC, AICC, BIC and HQIC values of models fitted for aircraft windshield dataset

Model	Negative LL	AIC	CAIC	BIC	HQIC	P-VALUE
E-LBE (Proposed Model)	126.9004	260.1845	261.0147	270.9546	264.7480	0.9251659
E-E	139.8405	285.681	285.981	292.9735	288.6125	0.1711434
E-L	140.4336	288.8672	289.3736	298.5905	292.7759	0.1659989
E-W	127.9121	263.8243	264.3306	273.5476	267.7330	0.8596897

From Table 6, the E-LBE has the highest log-likelihood values of -126.9004 and the lowest AIC, CAIC, BIC and HQIC values of 260.1845, 261.0147, 270.9546, and 264.7480 respectively. Hence, it is chosen as the best model amongst the considered models.

5 Conclusion

The E-LBE distribution has been successfully introduced in this paper and some of its basic statistical properties have been obtained. The pdf of the model and its failure rate have unimodal shapes; this means that the model would be useful to fit real-life events with unimodal failure rates. The model is tractable and flexible and shows high modeling capability as it performs better than the Exponentiated exponential, Exponentiated Lomax, and Exponentiated Weibull distributions; this was judged based on the AIC, CAIC, BIC, NLL and HQIC values of these distributions. The E-LBE distribution is of no doubt a competitive model, and it is hoped that it would be of use in fields like engineering, biology and medicine. Some other statistical properties of the distribution (which were not considered in this paper) can be explored and simulation study can also be conducted.

References

1. Abouammoh, A. M. & A. M. Alshingiti (2009). Reliability estimation of generalized inverted exponential distribution. Journal of Statistical Computation and Simulation 79 (11), 1301-1315.
2. Maxwell O, Friday AI, Chukwudike NC, et al. (2019). A theoretical analysis of the odd generalized exponentiated inverse Lomax distribution. Biom Biostat Int J. 2019; 8(1): 17-22. DOI: 10.15406/bbij.2019.08.00264.
3. Ristic, M..M., Kundu, D. (2015). Marshall-Olkin generalized exponential distribution, Metron 73(3), 317-333.
4. Aq, M.A.u., et al. Journal of King Saud University – Science (2017), http://dx.doi.org/10.1016/j.jksus.2017.09.06
5. Gupta, R.D., Kundu, D. (1999). Theory & methods: generalized exponential distributions. Australian & New Zealand J. Stat. 41 (2), 173–188
6. Gupta, R.D., Kundu, D. (2001). Exponentiated exponential family: an alternative to gamma and Weibull distributions. Biometrical J. 43 (1), 117–130
7. Iqbal, Z., Hasnain, S.A., Salman, M., Ahmad, M., Hamedani, G. (2014). Generalized exponentiated moment exponential distribution. Pakistan J. Stat.
8. Abu-Youssef, S., Mohammed, B., Sief, M. (2015). An extended exponentiated exponential distribution and its properties. Int. J. Comput. Appl. 121 (5)
9. Pogány, T.K., Saboor, A., Provost, S. (2015). The Marshall-Olkin exponential Weibull distribution. Hacettepe J. Math. Stat. 44, 1579–1594
10. Hasnain, S.A., Iqbal, Z., Ahmad, M. (2015). On exponentiated moment exponential distribution. Pak. J. Statist. 31 (2), 267–280.
11. Dara, S.T., Ahmad, M. (2012). Recent Advances in Moment Distribution and Their Hazard Rates. Lap Lambert Academic Publishing GmbH KG.
12. Cordeiro, G. M., Ortega, E. M. M., and Cunha, D. C. C. (2013). The exponentiated generalized class of distributions. Journal of Data Science, 11:1–27.
13. Thiago A. N. de Andrade, Marcelo Bourguignon, Gauss M. Cordeiro. (2016). The exponentiated generalized extended exponential distribution. Journal of Data Science, 14: 393-414
14. B. Abdul-Monoem & H.F. Abdel-Hameed (2019). On Exponentiated Lomax Distribution. International Journal of Mathematical Archive. 3(5):1-7.
15. M. Pal, M.M. Ali, & J.Woo. (2014). Exponentiated Weibull Distribution. Statistica, anno LXVI, n.2.
16. A. Flaih, H. Elsalloukh, E. Mendi, M. Milanova. (2012).The exponentiated inverted Weibull distribution. Appl. Math. Inf. Sci, 6 (2):167-171
17. Merovci, F., Khaleel, M.A., Ibrahim, N.A., & Shitan, M. (2016). The beta type X distribution: properties with applications. SpringerPlus, 5, 697. Doi:10.1186/s40064-016-2271-9
18. Obubu Maxwell, Samuel Oluwafemi Oyamakin, Eghwerido Joseph Th. The Gompertz Length Biased Exponential Distribution and its application to Uncensored Data. Curr Tre Biosta & Biometr 1(3) -2018. CTBB.MS.ID.000111.
19. Maxwell O, Chukwudike NC, Bright OC. Modeling lifetime data with the odd generalized exponentiated inverse Lomax distribution. Biom Biostat Int J. 2019; 8(2): 39-42. DOI: 10.15406/bbij.2019.08.00268.
20. Morais, A. L., and Barreto-Souza, W. (2011). A compound class of Weibull and power series distributions. Computational Statistics and Data Analysis 55(3): 1410-1425.

