On Making an Informed Decision between Four Exponential-based Continuous Compound Distributions

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Abstract. This article proposes a new continuous lifetime model called the Gompertz Alpha Power Inverted Exponential (G-APIE) distribution, and compares its modelling strength between the Extended Exponential distribution, Exponential Weibull, and Exponentiated Lomax distribution. The proposed distribution was applied to three lifetime data and the best model determined based on the lowest criterion values.

Keywords: Alpha power, inverted exponential, order statistics, Gompertz generalized family of distribution, hazard functions.

1 Introduction

Most of the statistical analyses deal with real life data sets under appropriate selection of models. The model selection is also an important issue for reliable estimation of the model parameters. Recently, some generalizations of the exponential distribution are proposed for modeling lifetime data due to some interesting properties such as "lack of memory" [10]. Keller et al. [9] introduced the inverted exponential (IE) distribution with inverted bathtub hazard rate. This inverted exponential distribution was further studied extensively by Lin et al. [11] and applied in engineering and medicine by Oguntunde and Adejumo (2013), Oguntunde et al. [16], Yousof et al. (2015), Afify et al. [2], Anake et al. [4], Aryal and Yousof [5], Cordeiro et al. [7] and Pinho et al. [18]. Most compound distributions perform better than their baseline distributions in terms of flexibility when applied to real life data sets. For instance; the Beta-Exponential distribution has been confirmed to perform better when compared with the Exponential distribution by Jafari, Tahmasebi and Alizadeh [22]; the Beta-Nakagami distribution has been confirmed to be a better alternative to the Nakagami distribution when applied to real life data sets by Shittu and Adepoju [23]; the Kumaraswamy-Dagum distribution out-performed the Dagum distribution when applied to a data set on air conditioning system [24]; the Transmuted Exponential distribution was confirmed to provide a better fit than the Exponential distribution when applied to real life data sets by Owoloko et al. [25]; the Gompertz length biased exponential distribution performed better than the length biased exponential when applied to an uncensored dataset (Obubu and Oyamakin, 2019); the odd generalized exponentiated inverse Lomax distribution outperformed the Inverse Lomax distribution when applied to real life datasets (Obubu et al, 2019). Therefore, it becomes interesting that generalising well-known standard distributions can produce more robust distributions. To this end, this article seeks to compound the Gompertz and alpha power inverted exponential distribution. The newly compounded distribution is further applied to carbon fibre and Aircraft Windshield dataset to examine its modelling strength in comparison to existing distributions.

Mahdavi and Kundu [12] proposed a G family of distributions called the alpha power with cdf given by

$$F_{AP}(x) = \frac{\alpha^{G(x)-1}}{\alpha-1} \text{ for } \alpha > 0, \alpha \neq 1 \tag{1}$$

with its pdf expressed as:

$$f_{AP}(x) = \frac{c \log \alpha}{x^2(\alpha-1)} g(x)\alpha^{G(x)} \text{ for } \alpha > 0, \alpha \neq 1 \tag{2}$$

where $g(x)$ and $G(x)$ are the baseline pdf and cdf respectively. Unal et al. [21] studied the case when g(x) and G(x) correspond to the inverted exponential distribution and gave the cdf and pdf of the alpha power inverted exponential distribution as:

$$F(x) = \frac{\alpha^{\exp(-\frac{c}{x})} - 1}{\alpha - 1} \quad \text{for } \alpha > 0, \alpha \neq 1 \tag{3}$$

$$f(x) = \frac{c \log \alpha}{x^2(\alpha - 1)} \exp\left(-\frac{c}{x}\right) \alpha^{\exp\left(-\frac{c}{x}\right)} \quad \text{for } \alpha > 0, \alpha \neq 1 \tag{4}$$

Thus, we proposed a new generalization of the alpha power inverted exponential distribution called the Gompertz Alpha Power Inverted Exponential (G-APIE) distribution. The rest of the paper is structured as follows; in section 2, we develop the probability density function, cumulative distribution function, the survival and hazard functions for the proposed Gompertz Alpha Power Inverted Exponential distribution, we also displayed their respective plots for different parameter values. In section 3, we tested the modeling strength of the Gompertz alpha power inverted length biased distribution, by applying the model to real life datasets and comparing with existing distributions. The concluding remark is presented in section 4.

2 The Gompertz-Alpha Power Inverted Exponential Distribution

The cumulative distribution function of the G-APIE distribution is given by

$$F_{G-APIE}(x) = \frac{ac \log \alpha}{x^2(\alpha - 1)} \exp\left(-\frac{c}{x}\right) \alpha^{\exp\left(-\frac{c}{x}\right)} \left[\frac{\alpha - 1}{\alpha - \alpha^{\exp\left(-\frac{c}{x}\right)}}\right]^{b+1} \exp\left(\frac{a}{b}\left(1 - \left[\frac{\alpha - 1}{\alpha - \alpha^{\exp\left(-\frac{c}{x}\right)}}\right]^{b}\right)\right), \text{for } \alpha > 0, \alpha \neq 1, c, b > 0 \tag{5}$$

The corresponding probability density function is given by

$$F_{G-APIE}(x) = 1 - \exp\left(\frac{a}{b}\left(1 - \left[\frac{\alpha - 1}{\alpha - \alpha^{\exp\left(-\frac{c}{x}\right)}}\right]^{b}\right)\right) \quad \text{for } \alpha > 0, \alpha \neq 1, c, b > 0 \tag{6}$$

The survival function is given by

$$S_{G-APIE}(x) = \exp\left(\frac{a}{b}\left(1 - \left[\frac{\alpha - 1}{\alpha - \alpha^{\exp\left(-\frac{c}{x}\right)}}\right]^{b}\right)\right), \quad \text{for } \alpha > 0, \alpha \neq 1, b, c > 0 \tag{7}$$

The hazard, reverse hazard and cumulative hazard function is given by equation (8), (9), (10) respectively

$$h_{G-APIE}(x) = \frac{ac \log \alpha}{x^2(\alpha - 1)} \exp\left(-\frac{c}{x}\right) \alpha^{\exp\left(-\frac{c}{x}\right)} \left[\frac{\alpha - 1}{\alpha - \alpha^{\exp\left(-\frac{c}{x}\right)}}\right]^{b+1}, \quad \text{for } \alpha > 0, \alpha \neq 1, b, c > 0 \tag{8}$$

$$H_{G-APIE}(x) = -\frac{a}{b}\left(1 - \left[\frac{\alpha - 1}{\alpha - \alpha^{\exp\left(-\frac{c}{x}\right)}}\right]^{b}\right), \quad \text{for } \alpha > 0, \alpha \neq 1, b, c > 0 \tag{9}$$

The plots for different parameter values of the G-AIPE distribution are given in figure 1-4 below

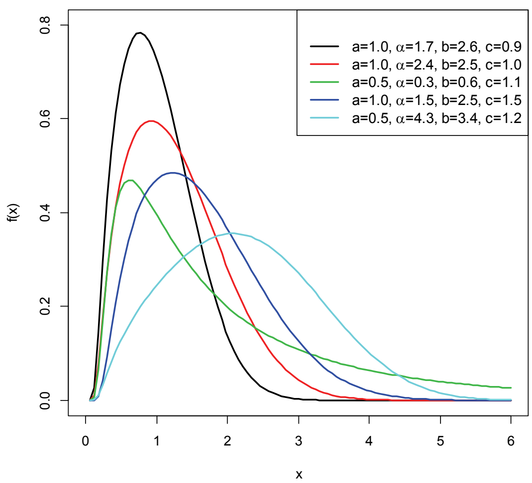

Figure 1. The pdf of the G-APIE distribution for different parameter values

Remark 1: It is clear in Figure 1 that the shape of the probability density function of the G-APIE distribution could be decreasing or inverted bathtub (depending on the value of the parameters). Also, it is positively skewed and has only one mode.

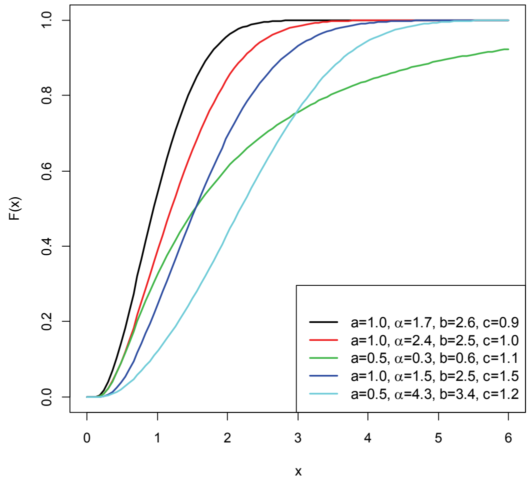

Figure 2. The cdf of the G-APIE distribution for different parameter values

Remark 2: The cumulative distribution plot of the G-APIE distribution clearly shows that the $x \to \infty$ the cdf of the G-APIE distribution tends to 1, suggesting that the G-APIE is a proper distribution.

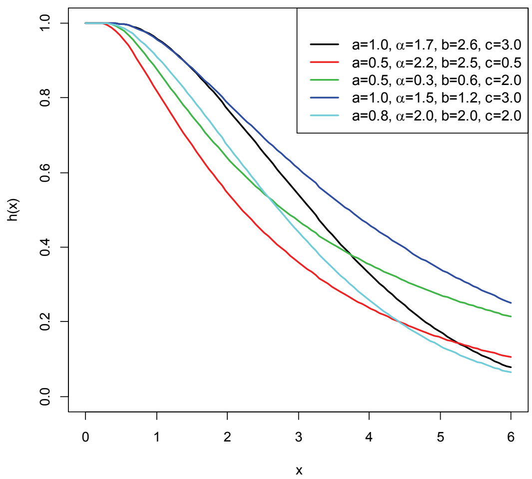

Figure 3. The survival function of the G-APIE distribution for different parameter values

Remark 3: Here we see that the probability of surviving past time zero is 1, and the survival curve goes to zero as $t \to \infty$

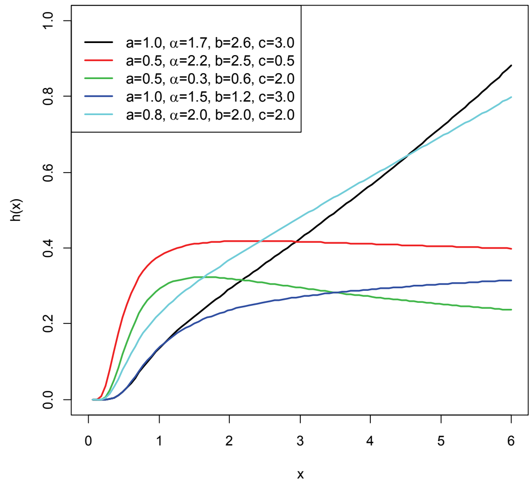

Figure 4. The hazard rate function of the G-APIE distribution for different parameter values

Remark 4: It can be deduced from Figure 4 that the shape of the hazard function of the G-APIE distribution could be increasing, or decreasing (depending on the value of the parameters)

3 Data Analysis

Here, the modeling strength of the Gompertz Alpha Power Inverse Exponential distribution is examined. To compare between fits of the Gompertz Alpha Power Inverse Exponential (G-APIE) distribution with that of Exponentiated Generalized Extended Exponential (E-E), Exponentiated-Lomax (E-L), and, Exponentiated-Weibull (E-W), two dataset were used in this study. The first being the tensile strength of 100 carbon fibers used by A. Flaih et al (2012) & Merovci, F et al [13], and the second being the Aircraft Windshield dataset reported be Morais, A. L., and Barreto-Souza, W. (2011). Criterion used to determine the distribution for the best fit include; The Akaike Information Criteria (AIC), Consistent Akaike Information Criteria (CAIC), Bayesian Information Criteria (BIC), and Hannan and Quinn Information Criteria (HQIC).

Data I: Tensile strength of carbon fibre

3.7, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 0.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22, 2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65.

Data II: Aircraft windshield data set

0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309,1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070,1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924, 1.281,2.038, 2.823, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432,2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506,2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619,2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757,2.324, 3.376, 4.663.

Table 1. Descriptive statistics for tensile strength of carbon fibre dataset

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Skewness	Kurtosis	Variance
0.39	1.830	2.675	2.601	3.197	5.560	0.3590264	3.126169	1.042162

It can be noticed from Table 1 that the data set is slightly positively skewed with the coefficient of skewness being 0.3590264 and a variance of 1.042162.

Table 2. Descriptive statistics for the aircraft windshield dataset

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Skewness	Kurtosis	Variance
0.040	1.839	2.354	2.557	3.393	4.663	0.09949403	2.347684	1.251768

It can be noticed from Table 2 that the data set is slightly positively skewed with the coefficient of Skewness being 0.09949403 and a variance of 1.251768

Table 3. Log-likelihood, AIC, AICC, BIC and HQIC values of models fitted for carbon fibre data

Model	Negative LL	AIC	CAIC	BIC	HQIC	RANK
G-APIE (Proposed Model)	141.3111	290.6222	291.0433	301.0429	294.8397	1
E-E	147.027	300.0541	300.3041	307.8696	303.2172	3
E-L	147.5959	303.1957	303.6168	313.6164	307.4132	4
E-W	142.0108	292.0215	292.4426	302.4422	296.239	2

Remark: The G-APIE has the highest log-likelihood values and the lowest AIC, CAIC, BIC and HQIC values; hence, it is the considered the best model amongst the competing models.

Table 4. Log-likelihood, AIC, AICC, BIC and HQIC values of models fitted for aircraft windshield dataset

Model	Negative LL	AIC	CAIC	BIC	HQIC	RANK
G-APIE (Proposed Model)	126.2154	261.0217	263.4152	269.1145	266.4141	1
E-E	139.8405	285.681	285.981	292.9735	288.6125	3
E-L	140.4336	288.8672	289.3736	298.5905	292.7759	4
E-W	127.9121	263.8243	264.3306	273.5476	267.7330	2

Remark: The G-APIE has the highest log-likelihood values and the lowest AIC, CAIC, BIC and HQIC values; hence, it is the considered the best model amongst the competing models.

4 Discussion

The performance of the G-APIE distribution with respect to the E-E distribution, E-L distribution and the E-W distribution using the observations in DATA I and DATA II as shown in Table 3 and Table 4 respectively. The distribution that corresponds to the lowest AIC, CAIC, BIC, HQIC or highest log-likelihood is considered the best fit. From Table 3, the G-APIE distribution has the highest log-likelihood value of -141.3111 and the lowest AIC value of 290.6222, CAIC value of 291.0433, BIC value of 301.0429, and HQIC value of 294.8397. And, from Table 4, the G-APIE distribution recorded the highest log-likelihood value of -126.2154 and the lowest AIC, CAIC, BIC, and HQIC value of 261.0217, 263.4152, 269.1145, and 266.4141 respectively. Therefore, the G-APIE distribution provides a better fit than the E-E, E-L and the E-W distributions.

5 Conclusion

The Gompertz-Alpha power inverted exponential (G-APIE) distribution has been successfully derived. The shape of the distribution could be inverted bathtub or decreasing (depending on the value of the parameters). An application to a two real life data shows that the Gompertz-Alpha power inverted exponential distribution is a better model for modelling when compared to the Exponentiated Exponential (E-E), Exponentiated Lomax (E-L), and the Exponentiated Weibull distribution.

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