A Generalised Approach on Generation of Commutative Matrix

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Abstract We proposed a generalised approach on generating commutative matrix of any non-singular matrix \( A \) (\( N \times N \)) satisfying the condition \([A, B_i] = 0 (i=1,2,3,4,......\infty)\)

Keywords: Inverse matrix, commutative matrix, continued fraction

Mathematical Classification (2010):16H15;14A05;05B20;15A8

1 Introduction

Matrix analysis is a powerful tool in understanding many feature of mathematics having direct relavance with physical systems. However it is commonly known that matrix multiplication has two important relations\[1\]

\[
\text{Det}(AB) = \text{Det}(A)\text{Det}(B) = \text{Det}(B)\text{Det}(A)
\]

\([A, B] \neq 0 \quad (2)\]

Interestingly it is possible to generate commutative matrices to a non-singular matrix\[2,3\]. In a recent paper any non-singular matrix \( A(NxN)(N=2,3) \) can possess commutative matrices \( B_L \) provided

\[
B_L = \frac{1}{L + A} \quad (3)
\]

Mathematically

\([A, B_L] = 0 \quad (4)\]

By varying \( L \) one can generate infinite no of commutative matrices \( B_L \). However in previous generation\[3\] ,the non-diagonal terms of entire \( B_L \) remain invariant with that of \( A \). Hence it is felt that one can generate new matrices having different diagonal and non-diagonal elements. The procedure is as follows.

2 Commutative Matrices

2.1 Commutative Matrices: Series

Here we suggest a procedure\[2-4\] to generate infinite matrices \( B_L \) as Let \( B_L \) is

\[
B_L = L + A + A^2 + A^3 + A^4 + \ldots \ldots = L + \sum_k A^k
\]

where \( k = 1,2,3,4,\ldots\ldots\infty \) Then it is easy to show that

\([A, B_L] = 0 \quad (6)\]

and

\([B, B_L] = 0 \quad (7)\]

From matrix theory \[1\] that one can have

\[A, B_L, A^{-1}, B_L^{-1} \to \Psi \quad (8)\]
Let
\[ A\psi = \alpha \psi \] (9)
and
\[ A^{-1}\psi = \frac{1}{\alpha} \psi \] (10)
\[ B_L\psi = \beta \psi \] (11)
and
\[ B_L^{-1}\psi = \frac{1}{\beta} \psi \] (12)
Then
\[ \alpha |\psi > = A|\psi > \] (13)
multiplying both sides by \( B_L^{-1} \) we have
\[ \alpha (B_L^{-1}|\psi >) = \frac{\alpha}{\beta} |\psi > = B_L^{-1}A|\psi > \] (14)
Similarly
\[ B_L^{-1}|\psi > = \frac{1}{\beta} |\psi > \] (15)
Multiplying \( A \) we have
\[ AB_L^{-1}|\psi > = \frac{1}{\beta} A|\psi > = \frac{\alpha}{\beta} |\psi > \] (16)
Hence we have
\[ AB^{-1} = B_L^{-1}A \rightarrow \frac{\alpha}{\beta} \] (17)
In other words
\[ [A, B_L^{-1}] = 0 \] (18)

2.2 Commutative Matrices: Continued Fraction

Here we select \( B \) as
\[ B_F = \frac{L}{L + \frac{A}{L + \frac{A}{L + \frac{A}{\ddots}}}} \] (19)
As in earlier case it is easy to show that
\[ [A, B_F^{-1}] = 0 \] (20)
Hence we have two sets of commutative matrices \( B_L^{-1} \) and \( B_F^{-1} \), corresponding to \( A \). Below we consider simple matrices and find out the form of \( B_L \) and \( B_F \) as follows.

3 Infinite Generation of Commutative Matrices (\( B_L \))

3.1 Infinite Generation of Commutative Matrices (\( B_L \)): Case Study (2x2)

Consider a simple (2x2) matrix \( A \) as [1-4]
\[ A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \] (21)

(i) \( B_1 = L + A \) as
\[ B_1 = L + A = \begin{bmatrix} 2 + L & 1 \\ 2 & 3 + L \end{bmatrix} \] (22)
Let us consider different values of \( L \) as follows. If \( L = 1 \) then we get the known matrix \([1]\) i.e.

\[
B_1 = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}
\]  

(23)

and

\[
B_1^{-1} = \begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}
\]  

(24)

Then it is easy to show that

\[
AB_1^{-1} = B_1^{-1}A = \begin{bmatrix} 0.6 & 0.1 \\ 0.2 & 0.7 \end{bmatrix}
\]  

(25)

\( (\text{ii}) B_2 = L + A + A^2 (L = 1) \)

In this case we get the known matrix \([1]\) i.e.

\[
B_2 = \begin{bmatrix} 9 & 6 \\ 12 & 15 \end{bmatrix}
\]  

(26)

and

\[
B_2^{-1} = \begin{bmatrix} 9 & -2 \\ 2 & 7 \end{bmatrix}
\]  

(27)

Then it is easy to show that

\[
AB_2^{-1} = B_2^{-1}A = \begin{bmatrix} 2 & -1 \\ 2 & 3 \end{bmatrix}
\]  

(28)

\( (\text{iii}) B_3 = L + A + A^2 + A^3 (L = 1) \)

In this case we get the known matrix \([1]\) i.e.

\[
B_3 = \begin{bmatrix} 31 & 27 \\ 54 & 58 \end{bmatrix}
\]  

(29)

and

\[
B_3^{-1} = \begin{bmatrix} 29 & -27 \\ 170 & 170 \end{bmatrix}
\]  

(30)

Then it is easy to show that

\[
AB_3^{-1} = B_3^{-1}A = \begin{bmatrix} 31 & -23 \\ -23 & 39 \end{bmatrix}
\]  

(31)

\( (\text{iv}) B_4 = L + A + A^2 + A^3 + A^4 (L = 1) \)

In this case we get the known matrix \([1]\) i.e.

\[
B_4 = \begin{bmatrix} 117 & 112 \\ 224 & 229 \end{bmatrix}
\]  

(32)

and

\[
B_4^{-1} = \begin{bmatrix} 229 & -112 \\ 1705 & 1705 \end{bmatrix}
\]  

(33)

Then it is easy to show that

\[
AB_4^{-1} = B_4^{-1}A = \begin{bmatrix} 234 & -107 \\ 1705 & 1705 \end{bmatrix}
\]  

(34)

\( (\text{v}) B_5 = L + A + A^2 + A^3 + A^4 + A^5 (L = 1) \)

In this case we get the known matrix \([1]\) i.e.

\[
B_5 = \begin{bmatrix} 459 & 453 \\ 906 & 912 \end{bmatrix}
\]  

(35)

and

\[
B_5^{-1} = \begin{bmatrix} 152 & -151 \\ 1365 & 1365 \end{bmatrix}
\]  

(36)

Then it is easy to show that

\[
AB_5^{-1} = B_5^{-1}A = \begin{bmatrix} 51 & -149 \\ -149 & 2730 \end{bmatrix}
\]  

(37)
3.2 Infinite Generation of Commutative Matrices ($B_L$): Case Study (3x3)

Here we just consider a simple example of (3x3) matrix and generate suitable commutative counter part as follows. The explicit expression for $A$ is

$$A = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -2 & 1 \\ -1 & 3 & 3 \end{bmatrix}$$

(38)

Considering the value of $L=1$ we get $B_5 = L + A + A^2 + A^3 + A^4 + A^5$ as

$$B_5 = \begin{bmatrix} 23 & 177 & 344 \\ 341 & 73 & 227 \\ -77491 & 858 \end{bmatrix}$$

(39)

$$B_5^{-1} = \begin{bmatrix} -227 & 4259 & 38 \\ \frac{16739}{1347} & \frac{88164}{1347} & \frac{88364}{1347} \\ \frac{1072}{2165} & \frac{14245}{2165} & \frac{14245}{2165} \end{bmatrix}$$

(40)

$$AB_5^{-1} = B_5^{-1}A = \begin{bmatrix} 163 & -331 & 64 \\ \frac{28362}{2165} & \frac{8371}{2165} & \frac{8371}{2165} \\ \frac{3525}{31459} & \frac{3683}{31459} & \frac{3683}{31459} \end{bmatrix}$$

(41)

4 Infinite generation of Commutative Matrices ($B_F$)

4.1 Infinite Generation of Commutative Matrices ($B_F$): Case Study (2x2)

Consider a simple (2x2) matrix $A$ as

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$

(42)

Now consider the matrix (i) $B_1 = L + A$ as

$$B_1 = L + A = \begin{bmatrix} 2 + L & 1 \\ 2 & 3 + L \end{bmatrix}$$

(43)

Let us consider different values of $L$ as follows $L=1$ In this case we get the known matrix [1] i.e

$$B_1 = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

(44)

and

$$B_1^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

(45)

Then it easy to show that

$$AB_1^{-1} = B_1^{-1}A = \begin{bmatrix} 8 & 6 \\ 12 & 14 \end{bmatrix}$$

(46)

Now consider the matrix (ii) $B_2 = L + A^\frac{1}{L+1}$ ($L=1$)

$$B_2 = \begin{bmatrix} 17 & -1 \\ -2 & 16 \end{bmatrix}$$

(47)

$$B_2^{-1} = \begin{bmatrix} 8 & 1 \\ 5 & 16 \end{bmatrix}$$

(48)
Then it easy to show that

\[ AB_2^{-1} = B_2^{-1}A = \begin{bmatrix} 17 & 19 \\ 15 & 13 \end{bmatrix} \]  \hspace{1cm} (49)

Now consider the matrix

(iii) \( B_3 = L + \frac{A}{L+\frac{A}{L+\frac{A}{L}}} (L=1) \)

\[ B_3 = \begin{bmatrix} \frac{73}{145} & -\frac{14}{145} \\ \frac{28}{145} & \frac{53}{145} \end{bmatrix} \]  \hspace{1cm} (50)

\[ B_3^{-1} = \begin{bmatrix} \frac{59}{27} & 14 \\ \frac{28}{27} & \frac{73}{27} \end{bmatrix} \]  \hspace{1cm} (51)

Then it easy to show that

\[ AB_3^{-1} = B_3^{-1}A = \begin{bmatrix} 146 & 101 \\ 272 & 27 \end{bmatrix} \]  \hspace{1cm} (52)

Now consider the matrix

(iv) \( B_4 = L + \frac{A}{L+\frac{A}{L+\frac{A}{L+\frac{A}{L}}} (L=1) \}

\[ B_4 = \begin{bmatrix} \frac{147}{260} & -\frac{31}{260} \\ \frac{29}{260} & \frac{29}{260} \end{bmatrix} \]  \hspace{1cm} (53)

\[ B_4^{-1} = \begin{bmatrix} \frac{243}{145} & \frac{31}{145} \\ \frac{139}{145} & \frac{139}{145} \end{bmatrix} \]  \hspace{1cm} (54)

Then it easy to show that

\[ AB_4^{-1} = B_4^{-1}A = \begin{bmatrix} 588 & 356 \\ 712 & 414 \end{bmatrix} \]  \hspace{1cm} (55)

Now consider the matrix

\[ B_5 = L + \frac{A}{L+\frac{A}{L+\frac{A}{L+\frac{A}{L}}} (L=1) \}

\[ B_5 = \begin{bmatrix} \frac{1247}{260} & -201 \\ \frac{29}{260} & \frac{29}{260} \end{bmatrix} \]  \hspace{1cm} (56)

\[ B_5^{-1} = \begin{bmatrix} \frac{523}{260} & 201 \\ \frac{29}{260} & \frac{29}{260} \end{bmatrix} \]  \hspace{1cm} (57)

Then it easy to show that

\[ AB_5^{-1} = B_5^{-1}A = \begin{bmatrix} 1247 & 1649 \\ 260 & 414 \end{bmatrix} \]  \hspace{1cm} (58)

### 4.2 Infinite Generation of Commutative Matrices \( (B_F) \): Case Study \((3x3)\)

Here we just consider a simple example of \((3x3)\) matrix[3] and generate suitable commutative counter part as follows. The explicit expression for \( A \) is [3]

\[ A = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -2 & 1 \\ -1 & 3 & 3 \end{bmatrix} \]  \hspace{1cm} (59)

Now consider the matrix \( B_5 = L + \frac{A}{L+\frac{A}{L+\frac{A}{L}}} (L=1) \)

\[ B_5 = \begin{bmatrix} \frac{932}{53} & -\frac{217}{53} & -\frac{781}{53} \\ \frac{1277}{53} & \frac{289}{53} & \frac{320}{53} \\ \frac{523}{53} & \frac{392}{53} & \frac{3140}{53} \end{bmatrix} \]  \hspace{1cm} (60)
\[ B_5^{-1} = \begin{bmatrix} 549 & 279 & 2479 \\ 240 & 1201 & 240 \\ 420 & 220 & 240 \end{bmatrix} \] (61)

\[ AB_5^{-1} = B_5^{-1} A = \begin{bmatrix} -1012 & 2411 & 2020 \\ -141 & -488 & 401 \\ 220 & 605 & 312 \end{bmatrix} \] (62)

5 Conclusion

This paper is the modified and more generalised version of the previous work of Rath[3] in generating commutative matrix of any non-singular matrix. In fact one can generate infinite set of commutative matrix. Further one can check that matrices generated in this paper are different from that generated in previous approach [3] using the known matrix \( A \). Here for the benefit of reader we have constructed considered simple cases \((N \times N) (N=2,3)\). However one can take any positive value of \( N \).

Acknowledgments. Author is thankful to Referee for constructive remarks.

References