Positive Solutions of Some Fourth-order Two Point Boundary Value Problem with All Order Derivatives

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Abstract In this paper, by the use of a new fixed point theorem and the Boundary Value Problem’s Green function, the existence of at least one positive solutions for the fourth-order two point boundary value problem with all order derivatives
\[
\begin{align*}
    u^{(4)}(t) + u''(t) &= \lambda f(t, u(t), u'(t), u''(t), u'''(t)), t \in [0, 1], \\
    u(0) &= u(1) = u''(0) = u''(1) = 0.
\end{align*}
\]

is considered, where \(f\) is a nonnegative continuous function and \(\lambda > 0, 0 < A < \pi^2\).

Keywords: Fourth-order boundary value problem, fixed point theorem in a cone, positive solution.

1 Introduction

The deformation of an elastic beam in equilibrium state, whose two ends are simply supported, can be described by a fourth-order ordinary equation boundary value problem. Owing to its significance in physics, the existence of positive solutions for the fourth-order boundary value problem has been studied by many authors using nonlinear alternatives of Leray-Schauder, the fixed point index theory, the Krasnosel’skii’s fixed point theorem and the method of upper and lower solutions, in reference [1-9][11].

In recent years, there has been much attention on the fourth-order differential equations with one or two parameters.

By the fixed point theorem and theory in cone [4], Bai investigated the following fourth-order two point boundary value problem
\[
\begin{align*}
    u^{(4)}(t) - \lambda f(u(t)) &= 0, t \in [0, 1], \\
    u(0) &= u(1) = u''(0) = u''(1) = 0.
\end{align*}
\]

where \(\lambda\) is a normal number, \(f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)\)

By the monotone operator theorem and the critical point theory, Li [7] proved the existence and multiplicities of positive solutions for the fourth-order two point boundary value problem
\[
\begin{align*}
    u^{(4)}(t) - f(u(t)) &= 0, t \in [0, 1], \\
    u(0) &= u(1) = u''(0) = u''(1) = 0.
\end{align*}
\]

where \(f : [0, 1] \times R^3 \rightarrow R^3\) is continuous.

All the above works were done under the assumption that the first order derivative \(u' u'' u'''\) is not involved explicitly in the nonlinear term \(f\). We are concerned with the existence of positive solutions for the fourth-order two-point boundary value problem
\[
\begin{align*}
    u^{(4)}(t) + u''(t) &= \lambda f(t, u(t), u'(t), u''(t), u'''(t)), t \in [0, 1], \\
    u(0) &= u(1) = u''(0) = u''(1) = 0.
\end{align*}
\] (1.1)

Throughout, we assume
\((H_1)\) \(\lambda > 0, 0 < A < \pi^2\);  
\((H_2)\) \(f : [0, 1] \times [0, \infty) \times R \rightarrow [0, \infty)\) is continuous.
2 Preliminary

Let $Y = C[0,1]$ be the Banach space equipped with the norm $\|u\|_0 = \max_{t \in [0,1]} |u(t)|$.

Set $\lambda_1, \lambda_2$ be the roots of the polynomial $P(\lambda) = \lambda^2 + A\lambda$, namely $\lambda_1 = 0, \lambda_2 = -A$. By $(H_1)$, it is easy to see that $-\pi^2 < \lambda_2 < 0$.

Let $G_i(t,s) (i = 1, 2)$ be the Green’s function of the linear boundary value problem: $-u'' + \lambda_i u(t) = 0, u(0) = u(1) = 0$. Then, carefully calculation yield:

$G_1(t,s) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\
t(1-s), & 0 \leq t \leq s \leq 1, \end{cases}$

$G_2(t,s) = \begin{cases} \frac{\sin \sqrt{A} \sin \sqrt{A} (1-t)}{\sqrt{A} \sin \sqrt{A}}, & 0 \leq s \leq t \leq 1, \\
\frac{\sin \sqrt{A} \sin \sqrt{A} (1-s)}{\sqrt{A} \sin \sqrt{A}}, & 0 \leq t \leq s \leq 1. \end{cases}$

Lemma 2.1. ([8]) Suppose $(H_1)(H_2)$ hold. Then for any $g(t) \in C[0,1]$, BVP

$$\begin{cases} u^{(4)}(t) + Au''(t) = g(t), t \in [0,1] \\
u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases}$$

has a unique solution

$$u(t) = \int_0^1 \int_0^1 G_1(t,s)G_2(s,\tau)g(\tau)d\tau ds, \quad (2.1)$$

where

$G_1(t,s) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\
t(1-s), & 0 \leq t \leq s \leq 1, \end{cases}$

$G_2(s,\tau) = \begin{cases} \frac{\sin \sqrt{A} \sin \sqrt{A} (1-s)}{\sqrt{A} \sin \sqrt{A}}, & 0 \leq \tau \leq s \leq 1, \\
\frac{\sin \sqrt{A} \sin \sqrt{A} (1-\tau)}{\sqrt{A} \sin \sqrt{A}}, & 0 \leq s \leq \tau \leq 1. \end{cases}$

By $u(t)$, we get

$$u'(t) = \int_0^1 \int_0^1 G_2(s,\tau)g(\tau)d\tau ds - \int_0^1 \int_0^1 sG_2(s,\tau)g(\tau)d\tau ds, \quad (2.2)$$

$$u''(t) = -\int_0^1 G_2(t,\tau)g(\tau)d\tau, \quad (2.3)$$

$$u'''(t) = -\int_0^1 \frac{\partial G_2(t,\tau)}{\partial \tau} g(\tau)d\tau. \quad (2.4)$$

Lemma 2.2. ([8]) Assume $(H_1)$ $(H_2)$ hold. Then one has:

(i) $G_i(t,s) \geq 0, \forall t, s \in [0,1]$;

(ii) $G_i(t,s) \leq C_1 G_i(s,s), \forall t, s \in [0,1]$;

(iii) $G_i(t,s) \geq \delta_1 G_i(t,t) G_i(s,s), \forall t, s \in [0,1]$.

where $C_1 = 1, \delta_1 = 1; C_2 = \frac{1}{\sin \sqrt{A}}, \delta_2 = \sqrt{A} \sin \sqrt{A}$.

Lemma 2.3. If $g(t) \in C[0,1], g(t) \geq 0$, then the unique solution $u(t)$ of the BVP (2.1) satisfies:

$$\min_{\frac{t_0}{4}}^{\frac{t_1}{4}} u(t) \geq d_1 \|u\|_0, \min_{\frac{t_0}{4}}^{\frac{t_1}{4}} (-u''(t)) \geq d_2 \|u''\|_0.$$
Proof. By (2.4) and (ii) of Lemma 2.2, we have

\[ u(t) = \int_0^1 \int_0^1 G_1(t,s)G_2(s,\tau)g(\tau)d\tau ds \]

\[ \leq C_1 C_2 \int_0^1 \int_0^1 G_1(s,s)G_2(\tau,\tau)g(\tau)d\tau ds \]

\[ \leq C_1 C_2 M_1 \int_0^1 G_2(\tau,\tau)g(\tau)d\tau \]

So,

\[ \|u(t)\| \leq C_1 C_2 M_1 \int_0^1 G_2(\tau,\tau)g(\tau)d\tau. \]

Using (iii) of Lemma 2.2, we have:

\[ u(t) \geq \delta_1 \delta_2 \int_0^1 \int_0^1 G_1(t,t)G_1(s,s)G_2(s,\tau)G_2(\tau,\tau)g(\tau)d\tau \]

\[ = \delta_1 \delta_2 C_0 G_1(t,t) \int_0^1 G_2(\tau,\tau)g(\tau)d\tau \]

\[ \geq \frac{\delta_1 \delta_2 C_0}{C_1 C_2 M_1} G_1(t,t)\|u(t)\|_0 \]

So,

\[ \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} u(t) \geq \frac{\delta_1 \delta_2 C_0 D_1}{C_1 C_2 M_1} \|u(t)\|_0 \]

\[ = \sqrt{A} \sin^2 \sqrt{A} C_0 D_1 \|u(t)\|_0 \]

\[ = d_1 \|u(t)\|_0. \]

By (2.6) and (ii) of Lemma 2.2, we have:

\[ -u''(t) = \int_0^1 G_2(t,\tau)g(\tau)d\tau \]

\[ \leq C_2 \int_0^1 G_2(\tau,\tau)g(\tau)d\tau \]

So, we have:

\[ \|u''(t)\|_0 = C_2 \int_0^1 G_2(\tau,\tau)g(\tau)d\tau. \]

Using (iii) of Lemma 2.2, We have:

\[ -u''(t) = \int_0^1 G_2(t,\tau)g(\tau)d\tau \]

\[ \geq \delta_2 \int_0^1 G_2(t,t)G_2(\tau,\tau)g(\tau)d\tau \]

\[ = \delta_2 G_2(t,t) \int_0^1 G_2(\tau,\tau)g(\tau)d\tau \]

\[ \geq \frac{\delta_2 G_2(t,t)}{C_2} \|u''(t)\|_0. \]
\[
\min_{\frac{1}{2} \leq i \leq \frac{3}{4}} (-u''(t)) \geq \frac{\delta_2 D_2}{C_2} \|u''(t)\|_0
\]
\[
= \sqrt{A} \sin^2 \sqrt{AD_2} \|u''(t)\|_0
\]
\[
= d_2 \|u''(t)\|_0
\]

Let \( X \) be a Banach space and \( K \subset X \) in a cone. Suppose \( \alpha, \beta : X \to R^+ \) are two continuous convex functionals satisfying \( \alpha(\lambda u) = |\lambda|\alpha(u), \beta(\lambda u) = |\lambda|\beta(u) \), for \( u \in X, \lambda \in R \), and \( |u| \leq M \max\{\alpha(u), \beta(u)\} \), for \( u \in X \) and \( \alpha(u) \leq \alpha(v) \) for \( u, v \in K, u \leq v \), where \( M > 0 \) is a constant.

**Theorem 2.1.** ([10]) Let \( r_2 > r_1 > 0, L > 0 \) be constants and

\[ \Omega_i = \{x \in X : \alpha(x) < r_i, \beta(x) < L\}, i = 1, 2, \]

two bounded open sets in \( X \). Set

\[ D_i = \{x \in X : \alpha(x) = r_i\}, i = 1, 2. \]

Assume \( T : K \to K \) is a completely continuous operator satisfying

\begin{align*}
(A_1) & \quad \alpha(Tx) < r_1, x \in D_1 \cap K; \alpha(Tx) > r_2, x \in D_2 \cap K; \\
(A_2) & \quad \beta(Tx) < L, x \in K; \\
(A_3) & \quad \text{there is a } p \in (\Omega_2 \cap K) \setminus \{0\} \text{ such that } \alpha(p) \neq 0 \text{ and } \alpha(x + \lambda p) \geq \alpha(x), \text{ for all } x \in K \text{ and } \lambda \geq 0.
\end{align*}

Then \( T \) has at least one fixed point in \( (\Omega_2 \setminus \Omega_1) \cap K \).

### 3 The main results

Let \( X = C^4[0, 1] \) be the Banach space equipped with the norm \( \|u\| = \max_{t \in [0, 1]} |u(t)| + \max_{t \in [0, 1]} |u'(t)| + \max_{t \in [0, 1]} |u''(t)| + \max_{t \in [0, 1]} |u'''(t)| \), and \( K = \{u \in X : u(t) \geq 0, u''(t) \leq 0, \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} (-u''(t)) \geq d_2 \|u''(t)\|_0\} \) is a cone in \( X \).

Define two continuous convex functionals \( \alpha(u) = \max_{t \in [0, 1]} |u(t)| + \max_{t \in [0, 1]} |u'(t)| + \max_{t \in [0, 1]} |u''(t)| \), and \( \beta(u) = \max_{t \in [0, 1]} |u'(t)| + \max_{t \in [0, 1]} |u''(t)| \), for each \( u \in X \), then \( \|u\| \leq 2 \max\{\alpha(u), \beta(u)\} \) and \( \alpha(\lambda u) = |\lambda|\alpha(x), \beta(\lambda u) = |\lambda|\beta(u) \), for \( u \in X, \lambda \in R; \alpha(u) \leq \alpha(v) \) for \( u, v \in K, u \leq v \).

In the following, we denote

\[ B = \int_0^1 G_2(\tau, \tau) d\tau, \]
\[ F = \int_0^1 \frac{\sin \sqrt{A} \tau}{\sin \sqrt{A}} d\tau, \]
\[ \eta_0 = \frac{1}{C_2 B(C_1 M_1 + 1)}, \eta_1 = \frac{1}{\int_0^\pi G_2(\tau, \tau) d\tau}, \eta_2 = \frac{2}{3C_2 B + 4F}, \theta = \{\frac{d_1}{2}, \frac{d_2}{2}\}. \]

We will suppose that there are \( L > b > \theta b > c > 0 \) such that \( f(t, u, v, u_0, v_0) \) satisfies the following growth conditions:

\( \begin{align*}
(H_3) & \quad f(t, u, v, u_0, v_0) < \frac{c\eta_0}{\lambda}, \text{ for } (t, u, v, u_0, v_0) \in [0, 1] \times [0, c] \times [-L, L] \times [-c, 0] \times [-L, L],
\end{align*} \)

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Then

We denote

\( H(t) \), for \((t, u, v, u_0, v_0) \in \left[ \frac{1}{4}, \frac{3}{2} \right] \times \left[ \theta b, b \right] \times [-L, L] \times [-b, 0] \times [-L, L] \) or \( [0, b] \times [-L, L] \times [-b, 0] \times [-L, L] \).

\( H_2 \) \( f(t, u, v, u_0, v_0) < \frac{L \eta}{\lambda} \), for \((t, u, v, u_0, v_0) \in [0, 1] \times [0, b] \times [-L, L] \times [-b, 0] \times [-L, L] \).

Let \( f_1(t, u, v, u_0, v_0) = f_1(t, u^*, v^*, u^*_0, v^*_0) \), where

\[
\begin{align*}
  u^* &= \max \{ \max(u, 0), b \}, \\
  v^* &= \max \{ \max(v, -L), L \}, \\
  u^*_0 &= \max \{ \max(u_0, 0), 0 \}, \\
  v^*_0 &= \max \{ \max(v_0, -L), L \}.
\end{align*}
\]

We denote

\[
(Tu)(t) = \lambda \int_0^1 \int_0^1 G_1(t, s)G_2(s, \tau)f_1(\tau, \tau), u^*(\tau), u^*(\tau), u^*(\tau))d\tau ds,
\]

\[
(Tu)'(t) = \lambda \int_0^1 \int_0^1 G_2(s, \tau)f_1(\tau, \tau), u^*(\tau), u^*(\tau), u^*(\tau))d\tau ds
\]

\[
- \int_0^1 \int_0^1 sG_2(s, \tau)f_1(\tau, \tau), u^*(\tau), u^*(\tau), u^*(\tau))d\tau ds,
\]

\[
(Tu)''(t) = -\lambda \int_0^1 G_2(t, \tau)f_1(\tau, \tau), u^*(\tau), u^*(\tau), u^*(\tau))d\tau,
\]

\[
(Tu)'''(t) = -\lambda \int_0^1 \frac{\partial G_2(t, \tau)}{\partial \tau}f_1(\tau, \tau), u^*(\tau), u^*(\tau), u^*(\tau))d\tau.
\]

**Lemma 3.1.** Suppose \((H_1)\) hold. Then \( T : K \to K \) is completely continuous. Suppose \((H_1) \) \((H_2)\) hold. Then \( T : K \to K \) is completely continuous.

**Proof.** For \( u \in K \), by (3.1) and (3.3) with Lemma 2.2, there is \( Tu > 0, (Tu)' \leq 0 \) so

\[
\begin{align*}
  \|Tu\|_0 &= \max_{t \in [0, 1]} \left| \lambda \int_0^1 \int_0^1 G_1(t, s)G_2(s, \tau)f_1(\tau, \tau), u^*(\tau), u^*(\tau), u^*(\tau))d\tau ds \right| \\
  &\leq \lambda \int_0^1 \int_0^1 C_1G_2(s, s)G_2(\tau, \tau)f_1(\tau, \tau), u^*(\tau), u^*(\tau), u^*(\tau))d\tau ds \\
  &= \lambda C_1C_2M_1 \int_0^1 G_2(\tau, \tau)f_1(\tau, \tau), u^*(\tau), u^*(\tau), u^*(\tau))d\tau,
\end{align*}
\]

\[
\begin{align*}
  \|(Tu)''\|_0 &= \max_{t \in [0, 1]} \left| -\lambda \int_0^1 G_2(t, \tau)f_1(\tau, \tau), u^*(\tau), u^*(\tau), u^*(\tau))d\tau \right| \\
  &\leq C_2 \int_0^1 G_2(\tau, \tau)f_1(\tau, \tau), u^*(\tau), u^*(\tau), u^*(\tau))d\tau
\end{align*}
\]

By Lemma 2.2, (ii) and (3.1) (3.3), we have:

\[
\begin{align*}
  \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} u(t) &= \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} \lambda \int_0^1 \int_0^1 G_1(t, s)G_2(s, \tau)f_1(\tau, \tau), u^*(\tau), u^*(\tau), u^*(\tau))d\tau ds \\
  &\geq \lambda \delta_1 \delta_2 \int_0^1 \int_0^1 G_1(t, t)G_2(s, s)G_2(\tau, \tau)f_1(\tau, \tau), u^*(\tau), u^*(\tau), u^*(\tau))d\tau d\tau \\
  &\geq \lambda \delta_1 \delta_2 C_0 \int_0^1 \int_0^1 G_2(\tau, \tau)f_1(\tau, \tau), u^*(\tau), u^*(\tau), u^*(\tau))d\tau d\tau \\
  &\geq \lambda \delta_1 \delta_2 C_0 D_1 \int_0^1 \int_0^1 G_2(\tau, \tau)f_1(\tau, \tau), u^*(\tau), u^*(\tau), u^*(\tau))d\tau d\tau \\
  &\geq \frac{\lambda \delta_1 \delta_2 C_0 D_1}{\lambda C_1 C_2 M_1} \|Tu\|_0 = d_1 \|Tu\|_0,
\end{align*}
\]
\[
\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} (-(Tu)''(t)) = \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} \lambda \int_{0}^{1} G_2(t, \tau) f_1(t, u(\tau), u'(\tau), u''(\tau), u'''(\tau)) d\tau \\
\geq \lambda \delta_2 \int_{0}^{1} G_2(t, t) G_2(\tau, \tau) f_1(t, u(\tau), u'(\tau), u''(\tau), u'''(\tau)) d\tau \\
\geq \lambda \delta_2 G_2(t, t) \int_{0}^{1} G_2(\tau, \tau) f_1(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)) d\tau \\
\geq \frac{\lambda \delta_2 G_2(t, t)}{C_2} \| (Tu)'' \|_0 \\
= d_2 \| (Tu)'' \|_0
\]

So we can get \( T(K) \subset K \). Let \( B \subset K \) is bounded, it is clear that \( T(B) \) is bounded. Using \( f_1, G_1(t, s), G_2(t, s) \) is continuous, We show that \( T(B) \) is equicontinuous. By the Arzela-Ascoli theorem, a standard proof yields \( T : K \rightarrow K \) is completely continuous.

\[\square\]

**Theorem 3.1.** Suppose \((H_1)-(H_5)\) hold. Then BVP (1.1) has at least one positive solution \( u(t) \) satisfying \( c < \alpha(u) < b \), \( \beta(u) < L \).

**Proof.** Take \( \Omega_1 = \{ u \in X : |\alpha(u)| < c, |\beta(u) - L| \} \), \( \Omega_2 = \{ u \in X : |\alpha(u)| < b, |\beta(u) - L| \} \), two bounded open sets in \( X \), and \( D_1 = \{ u \in X : |\alpha(u)| = c \}, D_2 = \{ u \in X : |\alpha(u)| = b \} \).

By Lemma 3.1, \( T : K \rightarrow K \) is completely continuous, and there is a \( p \in (\Omega_2 \cap K) \setminus \{0\} \) such that \( \alpha(p) \neq 0 \) for all \( u \in K \) and \( \lambda \geq 0 \).

\[
\| Tu \|_0 = | \lambda \int_{0}^{1} \int_{0}^{1} G_1(t, s) G_2(s, \tau) f_1(t, u(\tau), u'(\tau), u''(\tau), u'''(\tau)) ds d\tau |\\
\leq \lambda C_1 C_2 M_1 \int_{0}^{1} G_2(\tau, \tau) f_1(t, u(\tau), u'(\tau), u''(\tau), u'''(\tau)) d\tau \\
\leq \lambda C_1 C_2 M_1 \int_{0}^{1} G_2(\tau, \tau) d\tau \times \frac{c \eta_0}{\lambda} \\
= C_1 C_2 M_1 B c \eta_0,
\]

\[
\|(Tu)''\|_0 = | - \lambda \int_{0}^{1} G_2(t, \tau) f_1(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)) d\tau |\\
\leq \lambda C_2 \int_{0}^{1} G_2(\tau, \tau) d\tau \times \frac{c \eta_0}{\lambda} \\
= C_2 B c \eta_0.
\]

Hence, for \( u \in D_1 \cap K, \alpha(u) = c \), we get

\[
\alpha(Tu) = \| Tu \|_0 + \|(Tu)''\|_0 < C_1 C_2 M_1 B c \eta_0 + C_2 B c \eta_0 = (C_1 C_2 M_1 B + C_2 B) c \eta_0.
\]

Whereas for \( u \in D_2 \cap K, \alpha(u) = b \), there is \( \| u \|_0 \geq \frac{b}{2} \) or \( \| u'' \|_0 \geq \frac{b}{2} \). By Lemma 2.4, we get

\[
\min_{\frac{1}{4} \leq t \leq \frac{1}{4}} u(t) \geq d_1 \| u \|_0 \geq \frac{d_1 b}{2} \quad \text{or} \quad \min_{\frac{1}{4} \leq t \leq \frac{1}{4}} (-u''(t)) \geq \frac{d_2 c}{c_2} \| u'' \|_0 \geq \frac{d_2 b}{2}.
\]
Therefore, from \((H_4)\) and (3.3), we have

\[
| (Tu)^{'''} \left( \frac{1}{2} \right) | = | \lambda \int_0^1 G_2(\frac{1}{2}, \tau) f_1(\tau, u(\tau), u^{'}(\tau), u^{''}(\tau)) d\tau |
\]

\[
\geq \lambda \int_0^1 G_2(\frac{1}{2}, \tau) f_1(\tau, u(\tau), u^{'}(\tau), u^{''}(\tau)) d\tau
\]

\[
\geq \lambda \times \frac{bn_1}{A} \int_0^1 G_2(\frac{1}{2}, \tau) d\tau
\]

\[
= b.
\]

So,

\[
\alpha(Tu) \geq | (Tu)^{'''} \left( \frac{1}{2} \right) | = b.
\]

By (3.2) (3.4) and \((H_5)\), we have

\[
\| (Tu) \|_0 = \max_{t \in [0,1]} | \lambda \int_0^1 \int_0^1 G_2(s, \tau) f_1(\tau, u(\tau), u^{'}(\tau), u^{''}(\tau)) ds d\tau |
\]

\[
- \int_0^1 \int_0^1 s G_2(s, \tau) f_1(\tau, u(\tau), u^{'}(\tau), u^{''}(\tau)) ds d\tau |
\]

\[
< \max_{t \in [0,1]} | \lambda \int_0^1 \int_0^1 G_2(s, \tau) f_1(\tau, u(\tau), u^{'}(\tau), u^{''}(\tau)) ds d\tau |
\]

\[
+ \max_{t \in [0,1]} | \int_0^1 \int_0^1 s G_2(s, \tau) f_1(\tau, u(\tau), u^{'}(\tau), u^{''}(\tau)) ds d\tau |
\]

\[
\leq \lambda | \int_0^1 \int_0^1 (1 + s) G_2(s, \tau) f_1(\tau, u(\tau), u^{'}(\tau), u^{''}(\tau)) ds d\tau |
\]

\[
\leq \lambda \times \frac{\eta_2 L}{A} | \int_0^1 \int_0^1 (1 + s) G_2(s, \tau) ds d\tau |
\]

\[
\leq \frac{3C_2}{2} \eta_2 L \times | \int_0^1 G_2(\tau, \tau) d\tau |
\]

\[
= \frac{3C_2 B}{2} \eta_2 L,
\]

\[
\| (Tu) \|_0 = \max_{t \in [0,1]} | - \lambda \int_0^1 \int_0^1 \frac{\partial G_2(t, \tau)}{\partial \tau} f_1(t, u(\tau), u^{'}(\tau), u^{''}(\tau)) ds d\tau |
\]

\[
\leq \lambda \int_0^1 \frac{2 \sin \sqrt{A} \tau}{\sin \sqrt{A}} | f_1(t, u(\tau), u^{'}(\tau), u^{''}(\tau)) | d\tau
\]

\[
\leq \lambda \times \frac{\eta_2 L}{A} 2 \int_0^1 \frac{\sin \sqrt{A} \tau}{\sin \sqrt{A}} d\tau
\]

\[
= 2F \eta_2 L.
\]

Hence, for

\[
\beta(Tu) = \| (Tu) \|_0 + \| (Tu) \|_0 < \frac{3C_2 B}{2} \eta_2 L + 2F \eta_2 L < \left( \frac{3C_2 B}{2} + 2F \right) \eta_2 L = L.
\]

Theorem 2.1 implies there is \( u \in (\Omega_2 \setminus \Omega_1) \cap K \) such that \( u = Tu \). So, \( u(t) \) is a positive solution for BVP (1.1) satisfying

\[
c < \alpha(u) < b, \beta(u) < L.
\]

Thus, Theorem 3.1 is completed.
4 Conclusion

In this paper, the existence of at least one positive solutions for the fourth-order two point boundary value problem with all order derivatives is considered. By using a new cone fixed point theorem, the sufficient conditions for the existence of positive solutions of the boundary value problem are verified.

References