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Sharp Inequalities Involving Neuman Means of the Second Kind with Applications

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Abstract In this paper, we give the explicit formulas for Neuman means of the second kind $N_{GQ}(a,b)$ and $N_{QG}(a,b)$, and find the best possible parameters $\alpha_i, \beta_i \in (0,1) (i = 1, 2, 3, \cdots, 6)$ such that the double inequalities

$$\alpha_1 Q(a,b) + (1-\alpha_1)G(a,b) < N_{QG}(a,b) < \beta_1 Q(a,b) + (1-\beta_1)G(a,b),$$
$$\frac{\alpha_2}{G(a,b)} + \frac{1-\alpha_2}{Q(a,b)} < \frac{1}{N_{QG}(a,b)} < \frac{\beta_2}{G(a,b)} + \frac{1-\beta_2}{Q(a,b)},$$
$$\alpha_3 Q(a,b) + (1-\alpha_3)G(a,b) < N_{GQ}(a,b) < \beta_3 Q(a,b) + (1-\beta_3)G(a,b),$$
$$\frac{\alpha_4}{G(a,b)} + \frac{1-\alpha_4}{Q(a,b)} < \frac{1}{N_{GQ}(a,b)} < \frac{\beta_4}{G(a,b)} + \frac{1-\beta_4}{Q(a,b)},$$
$$\alpha_5 Q(a,b) + (1-\alpha_5)V(a,b) < N_{QG}(a,b) < \beta_5 Q(a,b) + (1-\beta_5)V(a,b),$$
$$\alpha_6 Q(a,b) + (1-\alpha_6)U(a,b) < N_{GQ}(a,b) < \beta_6 Q(a,b) + (1-\beta_6)U(a,b),$$

holds for all $a, b > 0$ with $a \neq b$, where $G(a,b)$ and $Q(a,b)$ are the classical geometric and quadratic means, $V(a,b)$, $U(a,b)$, $N_{QG}(a,b)$ and $N_{GQ}(a,b)$ are Yang and Neuman mean of the second kind.

Keywords: geometric mean, quadratic mean, Neuman means of the second kind, Yang means, inequalities.

1 Introduction

For $a, b > 0$ with $a \neq b$, the Schwab-Borchardt mean $SB(a,b)$[1,2] is defined by

$$SB(a,b) = \begin{cases} \dfrac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)}, & if\ a < b\ , \\ \dfrac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)}, & if\ a > b\ . \end{cases}$$

where $\cos^{-1}(x)$ and $\cosh^{-1}(x) = log(x + \sqrt{x^2-1})$ are the inverse cosine and inverse hyperbolic cosine functions, respectively.

It is well-known that $SB(a,b)$ is strictly increasing in both a and b, nonsymmetric and homogeneous of degree 1 with respect to a and b. Many symmetric bivariate means are special cases of the Schwab-Borchardt mean, for example, the first and second Seiffert means, Neuman-Sándor mean, logarithmic mean and two Yang means [3] are respectively defined by

$$P = P(a,b) = \frac{a-b}{2\sin^{-1}\left[(a-b)/(a+b)\right]} = SB(G, A)\ ,$$
$$T = T(a,b) = \frac{a-b}{2\tan^{-1}\left[(a-b)/(a+b)\right]} = SB(A, Q)\ ,$$

$$M = M(a,b) = \frac{a-b}{2\sinh^{-1}\big[(a-b)/(a+b)\big]} = SB(Q,A)\ ,$$
$$L = L(a,b) = \frac{a-b}{2\tanh^{-1}\big[(a-b)/(a+b)\big]} = SB(A,G)\ ,$$
$$U = U(a,b) = \frac{a-b}{\sqrt{2}\tan^{-1}\big[(a-b)/\sqrt{2ab}\big]} = SB(G,Q)\ , \tag{1}$$

and

$$V = V(a,b) = \frac{a-b}{\sqrt{2}\sinh^{-1}\big[(a-b)/\sqrt{2ab}\big]} = SB(Q,G)\ . \tag{2}$$

where $G = G(a,b) = \sqrt{ab}$, $A = A(a,b) = (a+b)/2$ and $Q = Q(a,b) = \sqrt{(a^2+b^2)/2}$ are the classical geometric, arithmetic and quadratic means of a and b.

Let $X = X(a,b)$ and $Y = Y(a,b)$ be the symmetric bivariate means of a and b. Then Neuman mean of the second kind $N_{XY}(a,b)$[4] is defined by

$$N_{XY}(a,b) = \frac{1}{2}\left[X + \frac{Y^2}{SB(X,Y)}\right]\ . \tag{3}$$

Moreover, without loss of generality, let $a > b$, $v = (a-b)/(a+b) \in (0,1)$, then Neuman [4] gave explicit formulas

$$N_{AG}(a,b) = \frac{1}{2}A\Big[1 + (1-v^2)\frac{tanh^{-1}(v)}{v}\Big], N_{GA}(a,b) = \frac{1}{2}A\Big[\sqrt{1-v^2} + \frac{sin^{-1}(v)}{v}\Big]$$
$$N_{AQ}(a,b) = \frac{1}{2}A\Big[1 + (1+v^2)\frac{tan^{-1}(v)}{v}\Big], N_{QA}(a,b) = \frac{1}{2}A\Big[\sqrt{1+v^2} + \frac{sinh^{-1}(v)}{v}\Big]$$

and inequalities

$$G(a,b) < L(a,b) < N_{AG}(a,b) < P(a,b) < N_{GA}(a,b) < A(a,b)$$
$$< M(a,b) < N_{QA}(a,b) < T(a,b) < N_{AQ}(a,b) < Q(a,b)\ .$$

for all $a, b > 0$ with $a \neq b$.

In the recent past, the Schwab-Borchardt mean has been the subject of intensive research. In particular, many remarkable inequalities for Schwab-Borchardt mean and its generated means can be found in the literature [4-14].

In [4], Neuman found the best possible constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\beta_1, \beta_2, \beta_3, \beta_4$ such that the double inequalities

$$\alpha_1 A(a,b) + (1-\alpha_1)G(a,b) < N_{GA}(a,b) < \beta_1 A(a,b) + (1-\beta_1)G(a,b)$$
$$\alpha_2 Q(a,b) + (1-\alpha_2)A(a,b) < N_{AQ}(a,b) < \beta_2 Q(a,b) + (1-\beta_2)A(a,b)$$
$$\alpha_3 A(a,b) + (1-\alpha_3)G(a,b) < N_{AG}(a,b) < \beta_3 A(a,b) + (1-\beta_3)G(a,b)$$
$$\alpha_4 Q(a,b) + (1-\alpha_4)A(a,b) < N_{QA}(a,b) < \beta_4 Q(a,b) + (1-\beta_4)A(a,b)$$

hold for $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 2/3$, $\beta_1 \geq \pi/4$, $\alpha_2 \leq 2/3$, $\beta_2 \geq (\pi-2)/\big[4(\sqrt{2}-1)\big] = 0.689\cdots$, $\alpha_3 \leq 1/3$, $\beta_3 \geq 1/2$ and $\alpha_4 \leq 1/3$, $\beta_4 \geq \big[\log(1+\sqrt{2}) + \sqrt{2} - 2\big]/\big[2(\sqrt{2}-1)\big] = 0.356\cdots$

Zhang et al. [11] presented the best possible parameters $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0, 1/2]$ and $\alpha_3, \alpha_4, \beta_3, \beta_4 \in [1/2, 1]$ such that the double inequalities

$$G\big(\alpha_1 a + (1-\alpha_1)b, \alpha_1 b + (1-\alpha_1)a\big) < N_{AG}(a,b) < G\big(\beta_1 a + (1-\beta_1)b, \beta_1 b + (1-\beta_1)a\big)$$
$$G\big(\alpha_2 a + (1-\alpha_2)b, \alpha_2 b + (1-\alpha_2)a\big) < N_{GA}(a,b) < G\big(\beta_2 a + (1-\beta_2)b, \beta_2 b + (1-\beta_2)a\big)$$
$$Q\big(\alpha_3 a + (1-\alpha_3)b, \alpha_3 b + (1-\alpha_3)a\big) < N_{QA}(a,b) < Q\big(\beta_3 a + (1-\beta_3)b, \beta_3 b + (1-\beta_3)a\big)$$
$$Q\big(\alpha_4 a + (1-\alpha_4)b, \alpha_4 b + (1-\alpha_4)a\big) < N_{AQ}(a,b) < Q\big(\beta_4 a + (1-\beta_4)b, \beta_4 b + (1-\beta_4)a\big)\ .$$

hold for all $a, b > 0$ with $a \neq b$.

Guo et.al. [12] proved that the double inequalities

$$\begin{gathered}
A^{p_1}(a,b)G^{1-p_1}(a,b) < N_{GA}(a,b) < A^{q_1}(a,b)G^{1-q_1}(a,b) \ , \\
\frac{p_2}{G(a,b)} + \frac{1-p_2}{A(a,b)} < N_{GA}(a,b) < \frac{q_2}{G(a,b)} + \frac{1-q_2}{A(a,b)} \ , \\
A^{p_3}(a,b)G^{1-p_3}(a,b) < N_{AG}(a,b) < A^{q_3}(a,b)G^{1-q_3}(a,b) \ , \\
\frac{p_4}{G(a,b)} + \frac{1-p_4}{A(a,b)} < N_{AG}(a,b) < \frac{q_4}{G(a,b)} + \frac{1-q_4}{A(a,b)} \ , \\
Q^{p_5}(a,b)A^{1-p_5}(a,b) < N_{AQ}(a,b) < Q^{q_5}(a,b)A^{1-q_5}(a,b) \ , \\
\frac{p_6}{A(a,b)} + \frac{1-p_6}{Q(a,b)} < N_{AQ}(a,b) < \frac{q_6}{A(a,b)} + \frac{1-q_6}{Q(a,b)} \ , \\
Q^{p_7}(a,b)A^{1-p_7}(a,b) < N_{QA}(a,b) < Q^{q_7}(a,b)A^{1-q_7}(a,b) \ , \\
\frac{p_8}{A(a,b)} + \frac{1-p_8}{Q(a,b)} < N_{QA}(a,b) < \frac{q_8}{A(a,b)} + \frac{1-q_8}{Q(a,b)} \ .
\end{gathered}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $p_1 \le 2/3$, $q_1 \ge 1$, $p_2 \le 0$, $q_2 \ge 1/3$, $p_3 \le 1/3$, $q_3 \ge 1$, $p_4 \le 0$, $q_4 \ge 2/3$, $p_5 \le 2/3$, $q_5 \ge 2\log(\pi+2)/\log 2 - 4 = 0.7244\cdots$, $p_6 \le \left[6+2\sqrt{2}-(1+\sqrt{2})\pi\right]/(\pi+2) = 0.2419\cdots$, $q_6 \ge 1/3$, $p_7 \le 1/3$, $q_7 \ge 2\log\left[\sqrt{2}+\log(1+\sqrt{2})\right]/\log 2 - 2 = 0.3977\cdots$ and $p_8 \le \left[2+\sqrt{2}-(1+\sqrt{2})\log(1+\sqrt{2})\right]/\left[\sqrt{2}+\log(1+\sqrt{2})\right] = 0.5603\cdots$, $q_8 \ge 2/3$.

Let $a > b > 0$, $u = (a-b)/\sqrt{2ab} \in (0, +\infty)$. Then from (1)-(3) we gave the explicit formulas

$$N_{QG}(a,b) = \frac{1}{2}G(a,b)\Big[\sqrt{1+u^2} + \frac{\sinh^{-1}(u)}{u}\Big] \ . \tag{4}$$

$$N_{GQ}(a,b) = \frac{1}{2}G(a,b)\Big[1 + (1+u^2)\frac{\tan^{-1}(u)}{u}\Big] \ . \tag{5}$$

The main purpose of this paper is to find the best possible parameters $\alpha_i, \beta_i \in (0,1)(i = 1, 2, 3, \cdots, 6)$ such that the double inequalities

$$\begin{gathered}
\alpha_1 Q(a,b) + (1-\alpha_1)G(a,b) < N_{QG}(a,b) < \beta_1 Q(a,b) + (1-\beta_1)G(a,b) \ , \\
\frac{\alpha_2}{G(a,b)} + \frac{1-\alpha_2}{Q(a,b)} < \frac{1}{N_{QG}(a,b)} < \frac{\beta_2}{G(a,b)} + \frac{1-\beta_2}{Q(a,b)} \ , \\
\alpha_3 Q(a,b) + (1-\alpha_3)G(a,b) < N_{GQ}(a,b) < \beta_3 Q(a,b) + (1-\beta_3)G(a,b) \ , \\
\frac{\alpha_4}{G(a,b)} + \frac{1-\alpha_4}{Q(a,b)} < \frac{1}{N_{GQ}(a,b)} < \frac{\beta_4}{G(a,b)} + \frac{1-\beta_4}{Q(a,b)} \ , \\
\alpha_5 Q(a,b) + (1-\alpha_5)V(a,b) < N_{QG}(a,b) < \beta_5 Q(a,b) + (1-\beta_5)V(a,b) \ , \\
\alpha_6 Q(a,b) + (1-\alpha_6)U(a,b) < N_{GQ}(a,b) < \beta_6 Q(a,b) + (1-\beta_6)U(a,b) \ .
\end{gathered}$$

hold for all $a, b > 0$ with $a \neq b$.

2 Lemma

In order to prove our main results we need several lemmas, which we present in this section.

Lemma 2.1 (see[15]) For $-\infty < a < b < +\infty$, let $f, g : [a, b] \to R$ be continuous on $[a, b]$, and be differentiable on (a, b), let $g'(x) \neq 0$ on (a, b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a, b), then so are

$$\frac{f(x)-f(a)}{g(x)-g(a)}, \quad \frac{f(x)-f(b)}{g(x)-g(b)}$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.2 (see [16]). Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ have the radius of convergence $r > 0$ with $a_n, b_n > 0$ for all $n = 0, 1, 2, \cdots$. Let $h(x) = f(x)/g(x)$, if the sequence series $\{a_n/b_n\}_{n=0}^{\infty}$ is (strictly) increasing (decreasing), then $h(x)$ is also (strictly) increasing (decreasing) on $(0, r)$.

Lemma 2.3 (1) (See [17], Lemma 2.4) The function

$$\varphi_1(x) = \frac{2x + \sinh(2x) - 4\sinh(x)}{\sinh(2x) - 2\sinh(x)}$$

is strictly increasing from $(0, +\infty)$ onto$(2/3, 1)$.

(2)(See [17], Lemma 2.6) The function

$$\varphi_2(x) = \frac{\sinh(x)\cosh(x) - x}{\big[\cosh(x) - 1\big]\big[x + \sinh(x)\cosh(x)\big]}$$

is strictly decreasing from $(0, +\infty)$ onto $(0, 2/3)$.

(3)(See [17], Lemma 2.5) The function

$$\varphi_3(x) = \frac{2x - \sin(2x)}{\sin(x)\big[1 - \cos(x)\big]}$$

is strictly increasing from $(0, \pi/2)$ onto$(8/3, \pi)$.

(4)(See [17], Lemma 2.8) The function

$$\varphi_4(x) = \frac{\sin(x)\cos(x) - x}{\big[1 - \cos(x)\big]\big[x + \sin(x)\cos(x)\big]}$$

is strictly decreasing from $(0, \pi/2)$ onto$(-1, -2/3)$.

Lemma 2.4 The function

$$\varphi_5(x) = \frac{x\sinh(2x) - 2x^2}{x\sinh(2x) - \cosh(2x) + 1}$$

is strictly decreasing from $(0, +\infty)$ onto$(1, 2)$.

Proof. Let $f_1(x) = x\sinh(2x) - 2x^2$, $g_1(x) = x\sinh(2x) - \cosh(2x) + 1$. Then simple computations lead to

$$\varphi_5(x) = \frac{f_1(x)}{g_1(x)} = \frac{f_1(x) - f_1(0^+)}{g_1(x) - g_1(0^+)} \ . \tag{6}$$

$$\begin{aligned} \frac{f_1'(x)}{g_1'(x)} &= \frac{\sinh(2x) + 2x\cosh(2x) - 4x}{2x\cosh(2x) - \sinh(2x)} \\ &= \frac{2x\sum_{n=0}^{\infty}\frac{2^{2n}}{(2n)!}x^{2n} + \sum_{n=0}^{\infty}\frac{2^{2n+1}}{(2n+1)!}x^{2n+1} - 4x}{2x\sum_{n=0}^{\infty}\frac{2^{2n}}{(2n)!}x^{2n} - \sum_{n=0}^{\infty}\frac{2^{2n+1}}{(2n+1)!}x^{2n+1}} \\ &= \frac{\sum_{n=1}^{\infty}\frac{(n+1)\times 2^{2n+2}}{(2n+1)!}x^{2n+1}}{\sum_{n=1}^{\infty}\frac{n\times 2^{2n+2}}{(2n+1)!}x^{2n+1}} = \frac{\sum_{n=0}^{\infty}\frac{(n+2)\times 2^{2n+4}}{(2n+3)!}x^{2n}}{\sum_{n=0}^{\infty}\frac{(n+1)\times 2^{2n+4}}{(2n+3)!}x^{2n}} \ . \end{aligned} \tag{7}$$

Let

$$a_n = \frac{(n+2)\times 2^{2n+4}}{(2n+3)!} > 0, b_n = \frac{(n+1)\times 2^{2n+4}}{(2n+3)!} > 0 \ . \tag{8}$$

and

$$\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = -\frac{1}{(n+1)(n+2)} < 0 \ . \tag{9}$$

for all $n \geq 0$.

It follows from Lemma 2.2 and (7)-(9) that $f_1'(x)/g_1'(x)$ is strictly decreasing on $(0, +\infty)$. Note that

$$\varphi_5(0^+) = \frac{a_0}{b_0} = 2, \varphi_5(+\infty) = 1 \ . \tag{10}$$

Therefore, Lemma 2.4 follows easily from Lemma 2.1 and (6), (10) together with the monotonicity of $f_1'(x)/g_1'(x)$.

Lemma 2.5 The function

$$\varphi_6(x) = \frac{x^2 + x\sin(x)\cos(x) - 2\sin^2(x)}{\sin(x)\big[x - \sin(x)\big]}$$

is strictly increasing from $(0, \pi/2)$ onto $(0, (\pi^2 - 8)/[2(\pi - 2)])$.

Proof. The function $\varphi_6(x)$ can be rewritten as

$$\varphi_6(x) = \frac{x}{\sin(x)} + \frac{x + x\cos(x) - 2\sin(x)}{x - \sin(x)} = \varphi_7(x) + \varphi_8(x) \ . \tag{11}$$

where $\varphi_7(x) = x/\sin(x)$ and $\varphi_8(x) = [x + x\cos(x) - 2\sin(x)]/[x - \sin(x)]$.

Let $f_2(x) = x+x\cos(x)-2\sin(x)$, $g_2(x) = x-\sin(x)$, $f_3(x) = 1-\cos(x)-x\sin(x)$ and $g_3(x) = 1-\cos(x)$. Then simple computations lead to

$$\varphi_8(x) = \frac{f_2(x)}{g_2(x)} = \frac{f_2(x) - f_2(0^+)}{g_2(x) - g_2(0^+)} \ . \tag{12}$$

$$\frac{f_2'(x)}{g_2'(x)} = \frac{f_3(x)}{g_3(x)} = \frac{f_3(x) - f_3(0^+)}{g_3(x) - g_3(0^+)} \ . \tag{13}$$

and

$$\frac{f_3'(x)}{g_3'(x)} = -\frac{x}{\tan(x)} \ . \tag{14}$$

Since the function $x \to x/\tan(x)$ is strictly decreasing on $(0, \pi/2)$, hence Lemma 2.1 and (12)-(14) lead to that $\varphi_8(x)$ is strictly increasing on $(0, \pi/2)$. From (11) and the fact that the function $\varphi_7(x) = x/\sin(x)$ is strictly increasing on $(0, \pi/2)$ together with the monotonicity of $\varphi_8(x)$ we can reach the conclusion that $\varphi_6(x)$ is strictly increasing on $(0, \pi/2)$.

Note that

$$\varphi_6(0^+) = 0, \varphi_6(\frac{\pi}{2}) = \frac{\pi^2 - 8}{2(\pi - 2)} \ . \tag{15}$$

Therefore, Lemma 2.5 follows easily from (15) and the monotonicity of $\varphi_6(x)$.

3 Main Results

Theorem 3.1 The double inequalities

$$\alpha_1 Q(a,b) + (1 - \alpha_1)G(a,b) < N_{QG}(a,b) < \beta_1 Q(a,b) + (1 - \beta_1)G(a,b) \ . \tag{16}$$

$$\frac{\alpha_2}{G(a,b)} + \frac{1 - \alpha_2}{Q(a,b)} < \frac{1}{N_{QG}(a,b)} < \frac{\beta_2}{G(a,b)} + \frac{1 - \beta_2}{Q(a,b)} \ . \tag{17}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 1/3$, $\beta_1 \geq 1/2$, $\alpha_2 \leq 0$ and $\beta_2 \geq 2/3$.

Proof. We clearly see that inequalities (16) and (17) can be rewritten as

$$\alpha_1 < \frac{N_{QG}(a,b) - G(a,b)}{Q(a,b) - G(a,b)} < \beta_1 \ . \tag{18}$$

and

$$\alpha_2 < \frac{1/N_{QG}(a,b) - 1/Q(a,b)}{1/G(a,b) - 1/Q(a,b)} < \beta_2 \ . \tag{19}$$

respectively.

Since both the geometric mean $G(a,b)$ and quadratic mean $Q(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b > 0$. Let $u = (a-b)/\sqrt{2ab} \in (0, +\infty)$. Then from (4) and (18)-(19) together with $Q(a,b) = G(a,b)\sqrt{1+u^2}$ we have

$$\alpha_1 < \frac{\frac{1}{2}\left[\sqrt{1+u^2} + \frac{\sinh^{-1}(u)}{u}\right] - 1}{\sqrt{1+u^2} - 1} < \beta_1 . \tag{20}$$

and

$$\alpha_2 < \frac{u\sqrt{1+u^2} - \sinh^{-1}(u)}{(\sqrt{1+u^2} - 1)\left[u\sqrt{1+u^2} + \sinh^{-1}(u)\right]} < \beta_2 . \tag{21}$$

respectively.

Let $x = \sinh^{-1}(u)$. Then $x \in (0, +\infty)$,

$$\begin{aligned} &\frac{\frac{1}{2}\left[\sqrt{1+u^2} + \frac{\sinh^{-1}(u)}{u}\right] - 1}{\sqrt{1+u^2} - 1} \\ &= \frac{1}{2}\frac{2x + \sinh(2x) - 4\sinh(x)}{\sinh(2x) - 2\sinh(x)} = \frac{1}{2}\varphi_1(x) . \end{aligned} \tag{22}$$

$$\begin{aligned} &\frac{u\sqrt{1+u^2} - \sinh^{-1}(u)}{(\sqrt{1+u^2} - 1)\left[u\sqrt{1+u^2} + \sinh^{-1}(u)\right]} \\ &= \frac{\sinh(x)\cosh(x) - x}{[\cosh(x) - 1][x + \sinh(x)\cosh(x)]} = \varphi_2(x) . \end{aligned} \tag{23}$$

where the functions $\varphi_1(x)$ and $\varphi_2(x)$ are defined as in Lemma 2.3(1) and (2).

Therefore, inequality (16) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 1/3$ and $\beta_1 \geq 1/2$ follows from (20) and (22) together with Lemma 2.3(1), inequality (17) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 \leq 0$ and $\beta_2 \geq 2/3$ follows from (21) and (23) together with Lemma 2.3(2).

Theorem 3.2 The double inequalities

$$\alpha_3 Q(a,b) + (1-\alpha_3)G(a,b) < N_{GQ}(a,b) < \beta_3 Q(a,b) + (1-\beta_3)G(a,b) . \tag{24}$$

$$\frac{\alpha_4}{G(a,b)} + \frac{1-\alpha_4}{Q(a,b)} < \frac{1}{N_{GQ}(a,b)} < \frac{\beta_4}{G(a,b)} + \frac{1-\beta_4}{Q(a,b)} . \tag{25}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_3 \leq 2/3$, $\beta_3 \geq \pi/4$, $\alpha_4 \leq 0$ and $\beta_4 \geq 1/3$.

Proof. We clearly see that inequalities (24) and (25) can be rewritten as

$$\alpha_3 < \frac{N_{GQ}(a,b) - G(a,b)}{Q(a,b) - G(a,b)} < \beta_3 . \tag{26}$$

and

$$\alpha_4 < \frac{1/N_{GQ}(a,b) - 1/Q(a,b)}{1/G(a,b) - 1/Q(a,b)} < \beta_4 . \tag{27}$$

respectively.

Since both the geometric mean $G(a,b)$ and quadratic mean $Q(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b > 0$. Let $u = (a-b)/\sqrt{2ab} \in (0, +\infty)$. Then from (5) and (26)-(27) together with $Q(a,b) = G(a,b)\sqrt{1+u^2}$ we have

$$\alpha_3 < \frac{\frac{1}{2}\left[1 + (1+u^2)\frac{\tan^{-1}(u)}{u}\right] - 1}{\sqrt{1+u^2} - 1} < \beta_3 . \tag{28}$$

and

$$\alpha_4 < \frac{2u\sqrt{1+u^2} - [u + (1+u^2)\tan^{-1}(u)]}{(\sqrt{1+u^2}-1)\Big[u + (1+u^2)\tan^{-1}(u)\Big]} < \beta_4 \ . \tag{29}$$

respectively.

Let $x = \tan^{-1}(u)$. Then $x \in (0, \pi/2)$,

$$\begin{aligned} &\frac{\frac{1}{2}\Big[1 + (1+u^2)\frac{\tan^{-1}(u)}{u}\Big] - 1}{\sqrt{1+u^2}-1} \\ &= \frac{1}{4}\frac{2x - \sin(2x)}{\sin(x)[1-\cos(x)]} = \frac{1}{4}\varphi_3(x) \ . \end{aligned} \tag{30}$$

$$\begin{aligned} &\frac{2u\sqrt{1+u^2} - [u + (1+u^2)\tan^{-1}(u)]}{(\sqrt{1+u^2}-1)\Big[u + (1+u^2)\tan^{-1}(u)\Big]} \\ &= 1 + \frac{\sin(x)\cos(x) - x}{[1-\cos(x)][x + \sin(x)\cos(x)]} = 1 + \varphi_4(x) \ . \end{aligned} \tag{31}$$

where the functions $\varphi_3(x)$ and $\varphi_4(x)$ are defined as in Lemma 2.3(3) and 2.3(4).

Therefore, inequality (24) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_3 \leq 2/3$ and $\beta_3 \geq \pi/4$ follows from (28) and (30) together with Lemma 2.3(3), inequality (25) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_4 \leq 0$ and $\beta_4 \geq 1/3$ follows from (29) and (31) together with Lemma 2.3(4).

Theorem 3.3 The double inequalities

$$\alpha_5 Q(a,b) + (1-\alpha_5)V(a,b) < N_{QG}(a,b) < \beta_5 Q(a,b) + (1-\beta_5)V(a,b) \ . \tag{32}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_5 \leq 0$ and $\beta_5 \geq 1/2$.

Proof. We clearly see that inequalities (32) can be rewritten as

$$\alpha_5 < \frac{N_{QG}(a,b) - V(a,b)}{Q(a,b) - V(a,b)} < \beta_5 \ . \tag{33}$$

Since both the geometric mean $G(a,b)$ and quadratic mean $Q(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b > 0$. Let $u = (a-b)/\sqrt{2ab} \in (0, +\infty)$. Then from (4) and (33) together with $Q(a,b) = G(a,b)\sqrt{1+u^2}$ we have

$$\alpha_5 < \frac{\frac{1}{2}\Big[\sqrt{1+u^2} + \frac{\sinh^{-1}(u)}{u}\Big] - \frac{u}{\sinh^{-1}(u)}}{\sqrt{1+u^2} - \frac{u}{\sinh^{-1}(u)}} < \beta_5 \ . \tag{34}$$

Let $x = \sinh^{-1}(u)$. Then $x \in (0, +\infty)$,

$$\begin{aligned} &\frac{\frac{1}{2}\Big[\sqrt{1+u^2} + \frac{\sinh^{-1}(u)}{u}\Big] - \frac{u}{\sinh^{-1}(u)}}{\sqrt{1+u^2} - \frac{u}{\sinh^{-1}(u)}} \\ &= 1 - \frac{1}{2}\frac{x\sinh(2x) - 2x^2}{x\sinh(2x) - \cosh(2x) + 1} = 1 - \frac{1}{2}\varphi_5(x) \ . \end{aligned} \tag{35}$$

where the functions $\varphi_5(x)$ is defined as in Lemma 2.4.

Therefore, inequality (32) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_5 \leq 0$ and $\beta_5 \geq 1/2$ follows from (34) and (35) together with Lemma 2.4.

Theorem 3.4 The double inequalities

$$\alpha_6 Q(a,b) + (1-\alpha_6)U(a,b) < N_{GQ}(a,b) < \beta_6 Q(a,b) + (1-\beta_6)U(a,b)\ . \tag{36}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_6 \leq 0$, $\beta_6 \geq (\pi^2-8)/[4(\pi-2)] = 0.4094\cdots$.

Proof. We clearly see that inequalities (36) can be rewritten as

$$\alpha_6 < \frac{N_{GQ}(a,b) - U(a,b)}{Q(a,b) - U(a,b)} < \beta_6\ . \tag{37}$$

Since both the geometric mean $G(a,b)$ and quadratic mean $Q(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b > 0$. Let $u = (a-b)/\sqrt{2ab} \in (0, +\infty)$. Then from (5) and (36) together with $Q(a,b) = G(a,b)\sqrt{1+u^2}$ we have

$$\alpha_6 < \frac{\frac{1}{2}\Big[1 + (1+u^2)\frac{\tan^{-1}(u)}{u}\Big] - \frac{u}{\tan^{-1}(u)}}{\sqrt{1+u^2} - \frac{u}{\tan^{-1}(u)}} < \beta_6\ . \tag{38}$$

Let $x = \tan^{-1}(u)$. Then $x \in (0, \pi/2)$,

$$\frac{\frac{1}{2}\Big[1 + (1+u^2)\frac{\tan^{-1}(u)}{u}\Big] - \frac{u}{\tan^{-1}(u)}}{\sqrt{1+u^2} - \frac{u}{\tan^{-1}(u)}} .= \frac{1}{2}\varphi_6(x), \tag{39}$$

where the function $\varphi_6(x)$ is defined as in Lemma 2.5.

Therefore, inequality (36) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_6 \leq 0$ and $\beta_6 \geq (\pi^2 - 8)/[4(\pi-2)] = 0.4094\cdots$ follows from (37)-(39) together with Lemma 2.5.

4 Applications

In this section, we will establish several sharp inequalities involving the hyperbolic, inverse hyperbolic, trigonometric and inverse trigonometric functions by use of Theorems 3.1-3.4.

From (3) we clearly see that

$$N_{QG}(a,b) = \frac{1}{2}\Big[Q(a,b) + \frac{G^2(a,b)}{V(a,b)}\Big],\ N_{GQ}(a,b) = \frac{1}{2}\Big[G(a,b) + \frac{Q^2(a,b)}{U(a,b)}\Big]. \tag{40}$$

Let $a > b$ and $x = \sinh^{-1}\left(\frac{a-b}{\sqrt{2ab}}\right) \in (0, \infty)$. Then simple computations lead to

$$\frac{Q(a,b)}{G(a,b)} = \cosh(x), \frac{V(a,b)}{G(a,b)} = \frac{\sinh(x)}{x}, \frac{U(a,b)}{G(a,b)} = \frac{\sinh(x)}{\tan^{-1}\big[\sinh(x)\big]}. \tag{41}$$

Theorems 3.1-3.4 and (40)-(41) lead to Theorem 4.1.

Theorem 4.1 The double inequalities

$$2\alpha_1\cosh(x) + 2(1-\alpha_1) < \cosh(x) + \frac{x}{\sinh(x)} < 2\beta_1\cosh(x) + 2(1-\beta_1),$$

$$\frac{1}{2}\big[\alpha_2\cosh(x) + (1-\alpha_2)\big] < 1 - \frac{2x}{\sinh(2x) + 2x} < \frac{1}{2}\big[\beta_2\cosh(x) + (1-\beta_2)\big],$$

$$2\alpha_3\cosh(x) + (1-2\alpha_3) < \cosh(x)\coth(x)\tan^{-1}\big[\sinh(x)\big] < 2\beta_3\cosh(x) + (1-2\beta_3),$$

$$\frac{\alpha_4\cosh(x) + (1-\alpha_4)}{2\cosh(x)} < \frac{1}{1 + \cosh(x)\coth(x)\tan^{-1}\big[\sinh(x)\big]} < \frac{\beta_4\cosh(x) + (1-\beta_4)}{2\cosh(x)},$$

$$2\alpha_5\cosh(x)+2(1-\alpha_5)\frac{\sinh(x)}{x}<\cosh(x)+\frac{x}{\sinh(x)}<2\beta_5\cosh(x)+2(1-\beta_5)\frac{\sinh(x)}{x},$$

$$2\alpha_6\cosh(x)+2(1-\alpha_6)\frac{\sinh(x)}{\tan^{-1}\big[\sinh(x)\big]}<1+\cosh(x)\coth(x)\tan^{-1}\big[\sinh(x)\big]$$
$$<2\beta_6\cosh(x)+2(1-\beta_6)\frac{\sinh(x)}{\tan^{-1}\big[\sinh(x)\big]}.$$

hold for all $x>0$ if and only if $\alpha_1\le 1/3$, $\beta_1\ge 1/2$, $\alpha_2\le 0$, $\beta_2\ge 2/3$, $\alpha_3\le 2/3$, $\beta_3\ge \pi/4$, $\alpha_4\le 0$, $\beta_4\ge 1/3$, $\alpha_5\le 0$, $\beta_5\ge 1/2$, $\alpha_6\le 0$ and $\beta_6\ge(\pi^2-8)/\big[4(\pi-2)\big]$.

Let $a>b$ and $x=\tan^{-1}\left(\frac{a-b}{\sqrt{2ab}}\right)\in(0,\pi/2)$. Then it is not difficult to verify that

$$\frac{Q(a,b)}{G(a,b)}=\sec(x),\ \frac{V(a,b)}{G(a,b)}=\frac{\tan(x)}{\sinh^{-1}\big[\tan(x)\big]},\ \frac{U(a,b)}{G(a,b)}=\frac{\tan(x)}{x}. \tag{42}$$

From Theorems 3.1-3.4 and (40), (42) we get Theorem 4.2 immediately.

Theorem 4.2 The double inequalities

$$2\alpha_1\sec(x)+2(1-\alpha_1)<\sec(x)+\frac{\sinh^{-1}\big[\tan(x)\big]}{\tan(x)}<2\beta_1\sec(x)+2(1-\beta_1),$$

$$\frac{1}{2}\big[\alpha_2+(1-\alpha_2)\cos(x)\big]<\frac{\tan(x)}{\sec(x)\tan(x)+\sinh^{-1}\big[\tan(x)\big]}<\frac{1}{2}\big[\beta_2+(1-\beta_2)\cos(x)\big],$$

$$2\alpha_3\sec(x)+2(1-\alpha_3)<1+\frac{2x}{\sin(2x)}<2\beta_3\sec(x)+2(1-\beta_3),$$

$$\frac{1}{2}\big[\alpha_4+(1-\alpha_4)\cos(x)\big]<1-\frac{2x}{\sin(2x)+2x}<\frac{1}{2}\big[\beta_4+(1-\beta_4)\cos(x)\big],$$

$$2\alpha_5\sec(x)+2(1-\alpha_5)\frac{\tan(x)}{\sinh^{-1}\big[\tan(x)\big]}<\sec(x)+\frac{\sinh^{-1}\big[\tan(x)\big]}{\tan(x)}<2\beta_5\sec(x)+2(1-\beta_5)\frac{\tan(x)}{\sinh^{-1}\big[\tan(x)\big]},$$

$$2\alpha_6\sec(x)+2(1-\alpha_6)\frac{\tan(x)}{x}<1+\frac{2x}{\sin(2x)}<2\beta_6\sec(x)+2(1-\beta_6)\frac{\tan(x)}{x}.$$

hold for all $x\in(0,\pi/2)$ if and only if $\alpha_1\le 1/3$, $\beta_1\ge 1/2$, $\alpha_2\le 0$, $\beta_2\ge 2/3$, $\alpha_3\le 2/3$, $\beta_3\ge\pi/4$, $\alpha_4\le 0$, $\beta_4\ge 1/3$, $\alpha_5\le 0$, $\beta_5\ge 1/2$, $\alpha_6\le 0$ and $\beta_6\ge(\pi^2-8)/\big[4(\pi-2)\big]$.

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