Sharp Inequalities Involving Neuman Means of the Second Kind with Applications

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Abstract. In this paper, we give the explicit formulas for Neuman means of the second kind \(N_{QG}(a, b)\) and \(N_{GQ}(a, b)\), and find the best possible parameters \(\alpha_i, \beta_i \in (0, 1) (i = 1, 2, 3, \ldots, 6)\) such that the double inequalities

\[
\begin{align*}
&\alpha_1 Q(a, b) + (1 - \alpha_1) G(a, b) < N_{QG}(a, b) < \beta_1 Q(a, b) + (1 - \beta_1) G(a, b), \\
&\frac{\alpha_2}{G(a, b)} + \frac{1 - \alpha_2}{Q(a, b)} < \frac{1}{N_{QG}(a, b)} < \frac{\beta_2}{G(a, b)} + \frac{1 - \beta_2}{Q(a, b)}, \\
&\alpha_3 Q(a, b) + (1 - \alpha_3) G(a, b) < N_{GQ}(a, b) < \beta_3 Q(a, b) + (1 - \beta_3) G(a, b), \\
&\frac{\alpha_4}{G(a, b)} + \frac{1 - \alpha_4}{Q(a, b)} < \frac{1}{N_{GQ}(a, b)} < \frac{\beta_4}{G(a, b)} + \frac{1 - \beta_4}{Q(a, b)}, \\
&\alpha_5 Q(a, b) + (1 - \alpha_5) V(a, b) < N_{GQ}(a, b) < \beta_5 Q(a, b) + (1 - \beta_5) V(a, b), \\
&\alpha_6 Q(a, b) + (1 - \alpha_6) U(a, b) < N_{GQ}(a, b) < \beta_6 Q(a, b) + (1 - \beta_6) U(a, b), 
\end{align*}
\]

holds for all \(a, b > 0\) with \(a \neq b\), where \(G(a, b)\) and \(Q(a, b)\) are the classical geometric and quadratic means, \(V(a, b), U(a, b), N_{QG}(a, b)\) and \(N_{GQ}(a, b)\) are Yang and Neuman mean of the second kind.

Keywords: geometric mean, quadratic mean, Neuman means of the second kind, Yang means, inequalities.

1 Introduction

For \(a, b > 0\) with \(a \neq b\), the Schwab-Borchardt mean \(SB(a, b)\) [1, 2] is defined by

\[
SB(a, b) = \left\{ \begin{array}{ll}
\sqrt{\frac{b^2 - a^2}{\cos^{-1}(a/b)}}, & \text{if } a < b, \\
\sqrt{\frac{a^2 - b^2}{\cosh^{-1}(a/b)}}, & \text{if } a > b.
\end{array} \right.
\]

where \(\cos^{-1}(x)\) and \(\cosh^{-1}(x) = \log(x + \sqrt{x^2 - 1})\) are the inverse cosine and inverse hyperbolic cosine functions, respectively.

It is well-known that \(SB(a, b)\) is strictly increasing in both \(a\) and \(b\), nonsymmetric and homogeneous of degree 1 with respect to \(a\) and \(b\). Many symmetric bivariate means are special cases of the Schwab-Borchardt mean, for example, the first and second Seiffert means, Neuman-Sándor mean, logarithmic mean and two Yang means [3] are respectively defined by

\[
P = P(a, b) = \frac{a - b}{2 \sin^{-1} [(a - b)/(a + b)]} = SB(G, A),
\]

\[
T = T(a, b) = \frac{a - b}{2 \tan^{-1} [(a - b)/(a + b)]} = SB(A, Q),
\]

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\[ M = M(a, b) = \frac{a - b}{2 \sinh^{-1} \left[ (a - b)/(a + b) \right]} = SB(Q, A) , \]
\[ L = L(a, b) = \frac{a - b}{2 \tanh^{-1} \left[ (a - b)/(a + b) \right]} = SB(A, G) , \]
\[ U = U(a, b) = \frac{a - b}{\sqrt{2} \tan^{-1} \left[ (a - b)/\sqrt{2ab} \right]} = SB(G, Q) , \]
\[ V = V(a, b) = \frac{a - b}{\sqrt{2} \sinh^{-1} \left[ (a - b)/\sqrt{2ab} \right]} = SB(Q, G) . \]

where \( G = G(a, b) = \sqrt{ab} , A = A(a, b) = (a + b)/2 \) and \( Q = Q(a, b) = \sqrt{(a^2 + b^2)/2} \) are the classical geometric, arithmetic and quadratic means of \( a \) and \( b \).

Let \( X = X(a, b) \) and \( Y = Y(a, b) \) be the symmetric bivariate means of \( a \) and \( b \). Then Neuman mean of the second kind \( N_{XY}(a, b) \)[4] is defined by
\[ N_{XY}(a, b) = \frac{1}{2} \left[ X + \frac{Y^2}{SB(X, Y)} \right] . \]

Moreover, without loss of generality, let \( a > b, v = (a - b)/(a + b) \in (0, 1) \), then Neuman [4] gave explicit formulas
\[ N_{AG}(a, b) = \frac{1}{2} A \left[ 1 + (1 - v^2) \frac{\tanh^{-1}(v)}{v} \right] , \]
\[ N_{QA}(a, b) = \frac{1}{2} A \left[ 1 + (1 + v^2) \frac{\tan^{-1}(v)}{v} \right] , \]
and inequalities
\[ G(a, b) < L(a, b) < N_{AG}(a, b) < P(a, b) < N_{GA}(a, b) < A(a, b) \]
\[ < M(a, b) < N_{QA}(a, b) < T(a, b) < N_{AQ}(a, b) < Q(a, b) . \]

for all \( a, b > 0 \) with \( a \neq b \).

In the recent past, the Schwab-Borchardt mean has been the subject of intensive research. In particular, many remarkable inequalities for Schwab-Borchardt mean and its generated means can be found in the literature [4-14].

In [4], Neuman found the best possible constants \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) and \( \beta_1, \beta_2, \beta_3, \beta_4 \) such that the double inequalities
\[ \alpha_1 A(a, b) + (1 - \alpha_1) G(a, b) < N_{GA}(a, b) < \beta_1 A(a, b) + (1 - \beta_1) G(a, b) \]
\[ \alpha_2 Q(a, b) + (1 - \alpha_2) A(a, b) < N_{AQ}(a, b) < \beta_2 Q(a, b) + (1 - \beta_2) A(a, b) \]
\[ \alpha_3 A(a, b) + (1 - \alpha_3) G(a, b) < N_{AG}(a, b) < \beta_3 A(a, b) + (1 - \beta_3) G(a, b) \]
\[ \alpha_4 Q(a, b) + (1 - \alpha_4) A(a, b) < N_{QA}(a, b) < \beta_4 Q(a, b) + (1 - \beta_4) A(a, b) \]
hold for \( a, b > 0 \) if and only if \( \alpha_1 \leq 2/3, \beta_1 \geq \pi/4, \alpha_2 \leq 2/3, \beta_2 \geq (\pi - 2)/[4(\sqrt{2} - 1)] = 0.689 \cdots, \alpha_3 \leq 1/3, \beta_3 \geq 1/2 \) and \( \alpha_4 \leq 1/3, \beta_4 \geq \left\lfloor \log(1 + \sqrt{2}) + \sqrt{2} - 2 \right\rfloor/[2(\sqrt{2} - 1)] = 0.356 \cdots \).

Zhang et al. [11] presented the best possible parameters \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in [0, 1/2] \) and \( \alpha_3, \alpha_4, \beta_3, \beta_4 \in [1/2, 1] \) such that the double inequalities
\[ G(\alpha_1 a + (1 - \alpha_1) b, \alpha_1 b + (1 - \alpha_1) a) < N_{AG}(a, b) < G(\beta_1 a + (1 - \beta_1) b, \beta_1 b + (1 - \beta_1) a) \]
\[ G(\alpha_2 a + (1 - \alpha_2) b, \alpha_2 b + (1 - \alpha_2) a) < N_{QA}(a, b) < G(\beta_2 a + (1 - \beta_2) b, \beta_2 b + (1 - \beta_2) a) \]
\[ Q(\alpha_3 a + (1 - \alpha_3) b, \alpha_3 b + (1 - \alpha_3) a) < N_{QA}(a, b) < Q(\beta_3 a + (1 - \beta_3) b, \beta_3 b + (1 - \beta_3) a) \]
\[ Q(\alpha_4 a + (1 - \alpha_4) b, \alpha_4 b + (1 - \alpha_4) a) < N_{AQ}(a, b) < Q(\beta_4 a + (1 - \beta_4) b, \beta_4 b + (1 - \beta_4) a) . \]
hold for all \(a, b > 0\) with \(a \neq b\).

Guo et.al. [12] proved that the double inequalities

\[
A(p, a, b)G(1-p)(a, b) < N_G(a, b) < A(q, a, b)G(1-q)(a, b),
\]

\[
\frac{p_3}{G(a, b)} + \frac{1 - p_3}{G(a, b)} < N_G(a, b) < \frac{q_3}{G(a, b)} + \frac{1 - q_3}{A(a, b)},
\]

\[
A(p, a, b)G(1-p)(a, b) < N_G(a, b) < A(q, a, b)G(1-q)(a, b),
\]

\[
\frac{p_4}{G(a, b)} + \frac{1 - p_4}{A(a, b)} < N_G(a, b) < \frac{q_4}{G(a, b)} + \frac{1 - q_4}{A(a, b)},
\]

\[
Q(p, a, b)A(1-p)(a, b) < N_Q(a, b) < Q(q, a, b)A(1-q)(a, b),
\]

\[
\frac{p_6}{A(a, b)} + \frac{1 - p_6}{Q(a, b)} < N_Q(a, b) < \frac{q_6}{A(a, b)} + \frac{1 - q_6}{Q(a, b)},
\]

\[
Q(p, a, b)A(1-p)(a, b) < N_Q(a, b) < Q(q, a, b)A(1-q)(a, b),
\]

\[
\frac{p_8}{A(a, b)} + \frac{1 - p_8}{Q(a, b)} < N_Q(a, b) < \frac{q_8}{A(a, b)} + \frac{1 - q_8}{Q(a, b)}.
\]

hold for all \(a, b > 0\) if and only if \(p_1 \leq 2/3, q_1 \geq 1, p_2 \leq 0, q_2 \geq 1/3, p_3 \leq 1/3, q_3 \geq 1, p_4 \leq 0, q_4 \geq 2/3, p_5 \leq 1/3, q_5 \leq 2 \log(p+2)/\log 2 - 4 = 0.7244 \cdots, p_6 \leq 6 + 2\sqrt{2} - (1 + \sqrt{2})/\pi + 2 = 0.2419 \cdots, q_6 \geq 1/3, p_7 \leq 1/3, q_7 \geq 2 \log(\sqrt{2} + \log(1 + \sqrt{2}))/\log 2 - 2 = 0.3977 \cdots \) and \(p_8 \leq 2 + 2\sqrt{2} - (1 + \sqrt{2}) \log(1 + \sqrt{2})/\sqrt{2} = 0.5603 \cdots, q_8 \geq 2/3.

Let \(a > b > 0\), \(u = (a-b)/\sqrt{2ab} \in (0, +\infty)\). Then from (1)-(3) we gave the explicit formulas

\[
N_Q(a, b) = \frac{1}{2} G(a, b) \left[ 1 + (1 + u^2)\frac{\sinh^{-1}(u)}{u} \right].
\]

(4)

\[
N_G(a, b) = \frac{1}{2} G(a, b) \left[ 1 + (1 + u^2)\tan^{-1}(u) \right].
\]

(5)

The main purpose of this paper is to find the best possible parameters \(\alpha_i, \beta_i \in (0, 1)(i = 1, 2, 3, \cdots, 6)\) such that the double inequalities

\[
\alpha_1 Q(a, b) + (1 - \alpha_1) G(a, b) < N_Q(a, b) < \beta_1 Q(a, b) + (1 - \beta_1) G(a, b),
\]

\[
\frac{\alpha_2}{Q(a, b)} + \frac{1 - \alpha_2}{Q(a, b)} < N_Q(a, b) < \frac{1}{Q(a, b)} + \frac{1 - \beta_2}{Q(a, b)},
\]

\[
\alpha_3 Q(a, b) + (1 - \alpha_3) G(a, b) < N_Q(a, b) < \beta_3 Q(a, b) + (1 - \beta_3) G(a, b),
\]

\[
\frac{\alpha_4}{Q(a, b)} + \frac{1 - \alpha_4}{Q(a, b)} < N_Q(a, b) < \frac{1}{Q(a, b)} + \frac{1 - \beta_4}{Q(a, b)},
\]

\[
\alpha_5 Q(a, b) + (1 - \alpha_5) V(a, b) < N_Q(a, b) < \beta_5 Q(a, b) + (1 - \beta_5) V(a, b),
\]

\[
\alpha_6 Q(a, b) + (1 - \alpha_6) U(a, b) < N_Q(a, b) < \beta_6 Q(a, b) + (1 - \beta_6) U(a, b).
\]

hold for all \(a, b > 0\) with \(a \neq b\).

2 Lemma

In order to prove our main results we need several lemmas, which we present in this section.

**Lemma 2.1** (sec[15]) For \(-\infty < a < b < +\infty\), let \(f, g : [a, b] \to R\) be continuous on \([a, b]\), and be differentiable on \((a, b)\), let \(g'(x) \neq 0\) on \((a, b)\). If \(f'(x)/g'(x)\) is increasing (decreasing) on \((a, b)\), then so are

\[
\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x) - f(b)}{g(x) - g(b)}
\]

If \(f'(x)/g'(x)\) is strictly monotone, then the monotonicity in the conclusion is also strict.
Lemma 2.2 (see [16]). Suppose that the power series \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) and \( g(x) = \sum_{n=0}^{\infty} b_n x^n \) have the radius of convergence \( r > 0 \) with \( a_n, b_n > 0 \) for all \( n = 0, 1, 2, \ldots \). Let \( h(x) = f(x)/g(x) \), if the sequence series \( \{a_n/b_n\}_{n=0}^{\infty} \) is (strictly) increasing (decreasing), then \( h(x) \) is also (strictly) increasing (decreasing) on \((0, r)\).

Lemma 2.3

1. (See [17], Lemma 2.4) The function
   \[
   \varphi_1(x) = \frac{2x + \sinh(2x) - 4 \sinh(x)}{\sinh(2x) - 2 \sinh(x)}
   \]
is strictly increasing from \((0, +\infty)\) onto \((2/3, 1)\).

2. (See [17], Lemma 2.6) The function
   \[
   \varphi_2(x) = \frac{\sinh(x) \cosh(x) - x}{[\cosh(x) - 1][x + \sinh(x) \cosh(x)]}
   \]
is strictly decreasing from \((0, +\infty)\) onto \((0, 2/3)\).

3. (See [17], Lemma 2.5) The function
   \[
   \varphi_3(x) = \frac{2x - \sin(2x)}{\sin(x)[1 - \cos(x)]}
   \]
is strictly increasing from \((0, \pi/2)\) onto \((8/3, \pi)\).

4. (See [17], Lemma 2.8) The function
   \[
   \varphi_4(x) = \frac{\sin(x) \cos(x) - x}{[1 - \cos(x)][x + \sin(x) \cos(x)]}
   \]
is strictly decreasing from \((0, \pi/2)\) onto \((-1, -2/3)\).

Lemma 2.4

The function
\[
\varphi_5(x) = \frac{x \sinh(2x) - 2x^2}{x \sinh(2x) - \cosh(2x) + 1}
\]
is strictly decreasing from \((0, +\infty)\) onto \((1, 2)\).

**Proof.** Let \( f_1(x) = x \sinh(2x) - 2x^2 \), \( g_1(x) = x \sinh(2x) - \cosh(2x) + 1 \). Then simple computations lead to
\[
\varphi_5(x) = \frac{f_1(x)}{g_1(x)} = \frac{f_1(x) - f_1(0^+)}{g_1(x) - g_1(0^+)}.
\]
\[
\frac{f_1'(x)}{g_1'(x)} = \frac{\sinh(2x) + 2x \cosh(2x) - 4x}{2x \cosh(2x) - \sinh(2x)}
\]
\[
= \frac{2x \sum_{n=0}^{\infty} \frac{2^n}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{2^{n+1}}{(2n+1)!} x^{2n+1} - 4x}{2x \sum_{n=0}^{\infty} \frac{2^n}{(2n)!} x^{2n} - \sum_{n=0}^{\infty} \frac{2^{n+1}}{(2n+1)!} x^{2n+1}}
\]
\[
= \frac{\sum_{n=1}^{\infty} \frac{(n+1) \times 2^{n+2}}{(2n+1)!} x^{2n+1} - \sum_{n=0}^{\infty} \frac{(n+2) \times 2^{n+4}}{(2n+3)!} x^{2n}}{\sum_{n=0}^{\infty} \frac{(n+1) \times 2^{n+4}}{(2n+3)!} x^{2n}}.
\]
Let
\[
a_n = \frac{(n+2) \times 2^{n+4}}{(2n+3)!} > 0, \quad b_n = \frac{(n+1) \times 2^{n+4}}{(2n+3)!} > 0.
\]
and
\[
a_{n+1} - a_n = \frac{1}{(n+1)(n+2)} < 0.
\]
for all $n \geq 0$.

It follows from Lemma 2.2 and (7)-(9) that $f'_1(x)/g'_1(x)$ is strictly decreasing on $(0, +\infty)$. Note that

$$\varphi_5(0^+) = \frac{a_0}{b_0} = 2, \varphi_5(+\infty) = 1.$$  \hspace{1cm} (10)

Therefore, Lemma 2.4 follows easily from Lemma 2.1 and (6), (10) together with the monotonicity of $f'_1(x)/g'_1(x)$.

**Lemma 2.5** The function

$$\varphi_6(x) = \frac{x^2 + x \sin(x) \cos(x) - 2 \sin^2(x)}{\sin(x)|x - \sin(x)|}$$

is strictly increasing from $(0, \pi/2)$ onto $(0, (\pi^2 - 8)/[2(\pi - 2)])$.

**Proof.** The function $\varphi_6(x)$ can be rewritten as

$$\varphi_6(x) = \frac{x}{\sin(x) + x \cos(x) - 2 \sin(x)} = \frac{x}{x - \sin(x)} \varphi_7(x) + \varphi_8(x),$$  \hspace{1cm} (11)

where $\varphi_7(x) = x/\sin(x)$ and $\varphi_8(x) = [x + x \cos(x) - 2 \sin(x)]/[x - \sin(x)]$.

Let $f_2(x) = x + x \cos(x) - 2 \sin(x), g_2(x) = x - \sin(x), f_3(x) = 1 - \cos(x) - x \sin(x)$ and $g_3(x) = 1 - \cos(x)$. Then simple computations lead to

$$\varphi_8(x) = f_2(x)/g_2(x) = f_2(x) - f_2(0^+) \quad g_2(x) - g_2(0^+),$$  \hspace{1cm} (12)

$$\frac{f'_2(x)}{g'_2(x)} = \frac{f_3(x) - f_3(0^+)}{g_3(x) - g_3(0^+)}.$$  \hspace{1cm} (13)

and

$$\frac{f'_3(x)}{g'_3(x)} = -\frac{x}{\tan(x).}$$  \hspace{1cm} (14)

Since the function $x \rightarrow x/\tan(x)$ is strictly decreasing on $(0, \pi/2)$, hence Lemma 2.1 and (12)-(14) lead to that $\varphi_8(x)$ is strictly increasing on $(0, \pi/2)$. From (11) and the fact that the function $\varphi_7(x) = x/\sin(x)$ is strictly increasing on $(0, \pi/2)$ together with the monotonicity of $\varphi_8(x)$ we can reach the conclusion that $\varphi_6(x)$ is strictly increasing on $(0, \pi/2)$.

Note that

$$\varphi_6(0^+) = 0, \varphi_6(\frac{\pi}{2}) = \frac{\pi^2 - 8}{2(\pi - 2)}.$$  \hspace{1cm} (15)

Therefore, Lemma 2.5 follows easily from (15) and the monotonicity of $\varphi_6(x)$.

**3 Main Results**

**Theorem 3.1** The double inequalities

$$\alpha_1 Q(a, b) + (1 - \alpha_1)G(a, b) < N_{QG}(a, b) < \beta_1 Q(a, b) + (1 - \beta_1)G(a, b).$$  \hspace{1cm} (16)

$$\frac{\alpha_2}{G(a, b)} + \frac{1 - \alpha_2}{Q(a, b)} < \frac{1}{N_{QG}(a, b)} < \frac{\beta_2}{G(a, b)} + \frac{1 - \beta_2}{Q(a, b)}.$$  \hspace{1cm} (17)

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 1/3, \beta_1 \geq 1/2, \alpha_2 \leq 0$ and $\beta_2 \geq 2/3$.

**Proof.** We clearly see that inequalities (16) and (17) can be rewritten as

$$\alpha_1 < \frac{N_{QG}(a, b) - G(a, b)}{Q(a, b) - G(a, b)} < \beta_1,$$  \hspace{1cm} (18)

and

$$\alpha_2 < \frac{1/N_{QG}(a, b) - 1/Q(a, b)}{1/G(a, b) - 1/Q(a, b)} < \beta_2.$$  \hspace{1cm} (19)
respectively.

Since both the geometric mean $G(a, b)$ and quadratic mean $Q(a, b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b > 0$. Let $u = (a - b)/\sqrt{2ab} \in (0, +\infty)$. Then from (4) and (18)-(19) together with $Q(a, b) = G(a, b)\sqrt{1 + u^2}$ we have

$$
\alpha_1 < \frac{1}{2} \left[ \sqrt{1 + u^2 + \frac{\sinh^{-1}(u)}{u}} \right] < \beta_1.
$$

and

$$
\alpha_2 < \frac{u\sqrt{1 + u^2} - \sinh^{-1}(u)}{(\sqrt{1 + u^2} - 1)\left[u\sqrt{1 + u^2} + \sinh^{-1}(u)\right]} < \beta_2.
$$

respectively.

Let $x = \sinh^{-1}(u)$. Then $x \in (0, +\infty)$,

$$
\frac{1}{2} \left[ \sqrt{1 + u^2 + \frac{\sinh^{-1}(u)}{u}} \right] - 1
= \frac{1}{2} \left[ \frac{2x + \sinh(2x) - 4\sinh(x)}{\sinh(2x) - 2\sinh(x)} \right] = \frac{1}{2} \varphi_1(x).
$$

Therefore, inequality (16) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 1/3$ and $\beta_1 \geq 1/2$ follows from (20) and (22) together with Lemma 2.3(1), inequality (17) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 \leq 0$ and $\beta_2 \geq 2/3$ follows from (21) and (23) together with Lemma 2.3(2).

**Theorem 3.2** The double inequalities

$$
\alpha_3 Q(a, b) + (1 - \alpha_3)G(a, b) < N_{GQ}(a, b) < \beta_3 Q(a, b) + (1 - \beta_3)G(a, b).
$$

$$
\alpha_4 \frac{G(a, b)}{Q(a, b)} + 1 - \alpha_4 < \frac{1}{N_{GQ}(a, b)} < \beta_4 \frac{G(a, b)}{Q(a, b)} + 1 - \beta_4
$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_3 \leq 2/3, \beta_3 \geq \pi/4, \alpha_4 \leq 0$ and $\beta_4 \geq 1/3$.

**Proof.** We clearly see that inequalities (24) and (25) can be rewritten as

$$
\alpha_3 < \frac{N_{GQ}(a, b) - G(a, b)}{Q(a, b) - G(a, b)} < \beta_3.
$$

and

$$
\alpha_4 < \frac{1/N_{GQ}(a, b) - 1/Q(a, b)}{1/G(a, b) - 1/Q(a, b)} < \beta_4.
$$

respectively.

Since both the geometric mean $G(a, b)$ and quadratic mean $Q(a, b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b > 0$. Let $u = (a - b)/\sqrt{2ab} \in (0, +\infty)$. Then from (5) and (26)-(27) together with $Q(a, b) = G(a, b)\sqrt{1 + u^2}$ we have

$$
\alpha_3 < \frac{1}{2} \left[ 1 + (1 + u^2)\tan^{-1}(u) \right] - 1
< \beta_3.
$$
and
\[
\alpha_4 < \frac{2u \sqrt{1 + u^2} - [u + (1 + u^2) \tan^{-1}(u)]}{(\sqrt{1 + u^2} - 1) [u + (1 + u^2) \tan^{-1}(u)]} < \beta_4 .
\] (29)
respectively.

Let \(x = \tan^{-1}(u)\). Then \(x \in (0, \pi/2)\),
\[
\frac{1}{2} \left[ 1 + \frac{(1 + u^2) \tan^{-1}(u)}{u} \right] - 1
\frac{\sqrt{1 + u^2} - 1}{2x - \sin(2x)} = \frac{1}{4} \sin(x) [1 - \cos(x)] = \frac{1}{4} \varphi_3(x).
\] (30)

Therefore, inequality (24) holds for all \(a, b > 0\) with \(a \neq b\) if and only if \(\alpha_3 \leq 2/3\) and \(\beta_3 \geq \pi/4\) follows from (28) and (30) together with Lemma 2.3(3), inequality (25) holds for all \(a, b > 0\) with \(a \neq b\) if and only if \(\alpha_4 \leq 0\) and \(\beta_4 \geq 1/3\) follows from (29) and (31) together with Lemma 2.3(4).

**Theorem 3.3** The double inequalities
\[
\alpha_5 Q(a, b) + (1 - \alpha_5) V(a, b) < N_{QG}(a, b) < \beta_5 Q(a, b) + (1 - \beta_5) V(a, b) .
\] (32)
holds for all \(a, b > 0\) with \(a \neq b\) if and only if \(\alpha_5 \leq 0\) and \(\beta_5 \geq 1/2\).

**Proof.** We clearly see that inequalities (32) can be rewritten as
\[
\alpha_5 < \frac{N_{QG}(a, b) - V(a, b)}{Q(a, b) - V(a, b)} < \beta_5 .
\] (33)

Since both the geometric mean \(G(a, b)\) and quadratic mean \(Q(a, b)\) are symmetric and homogeneous of degree 1, without loss of generality, we assume that \(a > b > 0\). Let \(u = (a - b)/\sqrt{2ab} \in (0, +\infty)\). Then from (4) and (33) together with \(Q(a, b) = G(a, b) \sqrt{1 + u^2}\) we have
\[
\alpha_5 < \frac{1}{2} \left[ \frac{\sqrt{1 + u^2} + \sinh^{-1}(u)}{u} - \frac{u}{\sinh^{-1}(u)} \right] < \beta_5 .
\] (34)

Let \(x = \sinh^{-1}(u)\). Then \(x \in (0, +\infty)\),
\[
\frac{1}{2} \left[ \frac{\sqrt{1 + u^2} + \sinh^{-1}(u)}{u} - \frac{u}{\sinh^{-1}(u)} \right]
\frac{\sqrt{1 + u^2} - \sinh^{-1}(u)}{\sinh^{-1}(u)}
= 1 - \frac{1}{2} \frac{x \sinh(2x) - 2x^2}{x \sinh(2x) - \cosh(2x) + 1} = 1 - \frac{1}{2} \varphi_5(x).
\] (35)

where the functions \(\varphi_5(x)\) is defined as in Lemma 2.4.

Therefore, inequality (32) holds for all \(a, b > 0\) with \(a \neq b\) if and only if \(\alpha_5 \leq 0\) and \(\beta_5 \geq 1/2\) follows from (34) and (35) together with Lemma 2.4.
Theorem 3.4 The double inequalities
\[ \alpha_6 Q(a, b) + (1 - \alpha_6)U(a, b) < N_{GQ}(a, b) < \beta_6 Q(a, b) + (1 - \beta_6)U(a, b). \] (36)
holds for all \( a, b > 0 \) with \( a \neq b \) if and only if \( \alpha_6 \leq 0, \beta_6 \geq (\pi^2 - 8)/[4(\pi - 2)] = 0.4094 \cdots \).

Proof. We clearly see that inequalities (36) can be rewritten as
\[ \alpha_6 < \frac{N_{GQ}(a, b) - U(a, b)}{Q(a, b) - U(a, b)} < \beta_6. \] (37)

Since both the geometric mean \( G(a, b) \) and quadratic mean \( Q(a, b) \) are symmetric and homogeneous of degree 1, without loss of generality, we assume that \( a > b > 0 \). Let \( u = (a - b)/\sqrt{2ab} \in (0, +\infty) \). Then from (5) and (36) together with \( Q(a, b) = G(a, b)\sqrt{1 + u^2} \) we have
\[ \alpha_6 < \frac{1}{2} \left[ 1 + (1 + u^2)\frac{\tan^{-1}(u)}{u} \right] - \frac{u}{\tan^{-1}(u)} < \beta_6. \] (38)

Let \( x = \tan^{-1}(u) \). Then \( x \in (0, \pi/2) \),
\[ \frac{1}{2} \left[ 1 + (1 + u^2)\frac{\tan^{-1}(u)}{u} \right] - \frac{u}{\tan^{-1}(u)} = \frac{1}{2} \varphi_6(x), \] (39)
where the function \( \varphi_6(x) \) is defined as in Lemma 2.5.

Therefore, inequality (36) holds for all \( a, b > 0 \) with \( a \neq b \) if and only if \( \alpha_6 \leq 0 \) and \( \beta_6 \geq (\pi^2 - 8)/[4(\pi - 2)] = 0.4094 \cdots \) follows from (37)-(39) together with Lemma 2.5.

4 Applications

In this section, we will establish several sharp inequalities involving the hyperbolic, inverse hyperbolic, trigonometric and inverse trigonometric functions by use of Theorems 3.1-3.4.

From (3) we clearly see that
\[ N_{GQ}(a, b) = \frac{1}{2} \left[ Q(a, b) + \frac{G^2(a, b)}{V(a, b)} \right], \quad N_{GQ}(a, b) = \frac{1}{2} \left[ G(a, b) + \frac{Q^2(a, b)}{U(a, b)} \right]. \] (40)

Let \( a > b \) and \( x = \sinh^{-1}\left(\frac{a-b}{\sqrt{2ab}}\right) \in (0, \infty) \). Then simple computations lead to
\[ \frac{Q(a, b)}{G(a, b)} = \cosh(x), \quad \frac{V(a, b)}{G(a, b)} = \frac{\sinh(x)}{x}, \quad \frac{U(a, b)}{G(a, b)} = \frac{\sinh(x)}{\tan^{-1}\left(\sinh(x)\right)}. \] (41)

Theorems 3.1-3.4 and (40)-(41) lead to Theorem 4.1.

Theorem 4.1 The double inequalities
\[ 2\alpha_1 \cosh(x) + 2(1 - \alpha_1) < \cosh(x) + \frac{x}{\sinh(x)} < 2\beta_1 \cosh(x) + 2(1 - \beta_1), \]
\[ \frac{1}{2} \left[ \alpha_2 \cosh(x) + (1 - \alpha_2) \right] < 1 - \frac{2x}{\sinh(2x) + 2x} < \frac{1}{2} \left[ \beta_2 \cosh(x) + (1 - \beta_2) \right], \]
\[ 2\alpha_3 \cosh(x) + (1 - 2\alpha_3) < \cosh(x) \coth(x) \tan^{-1}\left(\sinh(x)\right) < 2\beta_3 \cosh(x) + (1 - 2\beta_3), \]
\[ \frac{\alpha_4 \cosh(x) + (1 - \alpha_4)}{2 \cosh(x)} < \frac{1}{1 + \cosh(x) \coth(x) \tan^{-1}\left(\sinh(x)\right)} < \frac{\beta_4 \cosh(x) + (1 - \beta_4)}{2 \cosh(x)}, \]
\[ 2\alpha_5 \cosh(x) + 2(1 - \alpha_5) \frac{\sinh(x)}{x} < \cosh(x) + \frac{x}{\sinh(x)} < 2\beta_5 \cosh(x) + 2(1 - \beta_5) \frac{\sinh(x)}{x}, \]

\[ 2\alpha_6 \cosh(x) + 2(1 - \alpha_6) \frac{\sinh(x)}{\tan^{-1}[\sinh(x)]} < 1 + \cosh(x) \cot(x) \tan^{-1} \left[ \frac{\sinh(x)}{\tan^{-1}[\sinh(x)]} \right] < 2\beta_6 \cosh(x) + 2(1 - \beta_6) \frac{\sinh(x)}{\tan^{-1}[\sinh(x)]}, \]

hold for all \( x > 0 \) if and only if \( \alpha_1 \leq 1/3, \beta_1 \geq 1/2, \alpha_2 \leq 0, \beta_2 \geq 2/3, \alpha_3 \leq 2/3, \beta_3 \geq \pi/4, \alpha_4 \leq 0, \beta_4 \geq 1/3, \alpha_5 \leq 0, \beta_5 \geq 1/2, \alpha_6 \leq 0 \) and \( \beta_6 \geq (\pi^2-8)/[4(\pi - 2)] \).

Let \( a > b \) and \( x = \tan^{-1} \left( \frac{a-b}{\sqrt{2ab}} \right) \in (0, \pi/2) \). Then it is not difficult to verify that

\[ \frac{Q(a,b)}{G(a,b)} = \sec(x), \quad \frac{V(a,b)}{G(a,b)} = \frac{\tan(x)}{\sinh^{-1}[\tan(x)]}, \quad \frac{U(a,b)}{G(a,b)} = \frac{\tan(x)}{x}. \]

From Theorems 3.1-3.4 and (40), (42) we get Theorem 4.2 immediately.

**Theorem 4.2** The double inequalities

\[ 2\alpha_1 \sec(x) + 2(1 - \alpha_1) < \sec(x) + \frac{\sinh^{-1}[\tan(x)]}{\tan(x)} < 2\beta_1 \sec(x) + 2(1 - \beta_1), \]

\[ \frac{1}{2} \left[ \alpha_2 + (1 - \alpha_2) \cos(x) \right] < \tan(x) \frac{\tan(x)}{\sec(x) \tan(x) + \sinh^{-1}[\tan(x)]} < \frac{1}{2} \left[ \beta_2 + (1 - \beta_2) \cos(x) \right], \]

\[ 2\alpha_3 \sec(x) + 2(1 - \alpha_3) < 1 + \frac{2x}{\sin(2x)} < 2\beta_3 \sec(x) + 2(1 - \beta_3), \]

\[ \frac{1}{2} \left[ \alpha_4 + (1 - \alpha_4) \cos(x) \right] < 1 - \frac{2x}{\sin(2x) + 2x} < \frac{1}{2} \left[ \beta_4 + (1 - \beta_4) \cos(x) \right], \]

\[ 2\alpha_5 \sec(x) + 2(1 - \alpha_5) \frac{\tan(x)}{\sinh^{-1}[\tan(x)]} < \sec(x) + \frac{\sinh^{-1}[\tan(x)]}{\tan(x)} < 2\beta_5 \sec(x) + 2(1 - \beta_5) \frac{\tan(x)}{\sinh^{-1}[\tan(x)]}, \]

\[ 2\alpha_6 \sec(x) + 2(1 - \alpha_6) \frac{\tan(x)}{x} < 1 + \frac{2x}{\sin(2x)} < 2\beta_6 \sec(x) + 2(1 - \beta_6) \frac{\tan(x)}{x}. \]

hold for all \( x \in (0, \pi/2) \) if and only if \( \alpha_1 \leq 1/3, \beta_1 \geq 1/2, \alpha_2 \leq 0, \beta_2 \geq 2/3, \alpha_3 \leq 2/3, \beta_3 \geq \pi/4, \alpha_4 \leq 0, \beta_4 \geq 1/3, \alpha_5 \leq 0, \beta_5 \geq 1/2, \alpha_6 \leq 0 \) and \( \beta_6 \geq (\pi^2-8)/[4(\pi - 2)] \).

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**References**