Journal of Advances in Applied Mathematics, Vol. 1, No. 1, January 2016
https://dx.doi.org/10.22606/jaam.2016.11001

Antenna Synthesis by the Modulus of the Diagram

Alexander G. Ramm[1], Mykhaylo I. Andriychuk[2*]

[1] Mathematics Department, Kansas State University, Manhattan, Kansas, United States
[2] Institute for Applied Problems of Mechanics and Mathematics, NASU, Lviv, Ukraine
Email: andr@iapmm.lviv.ua

Abstract. A nonlinear antenna synthesis problem is considered. Given a function that approximates the modulus of the diagram, an explicit formula for the corresponding current is found. This current generates the diagram as close to the given function as one wants. The numerical results illustrating the developed method are presented. The results show that in order to achieve a desired accuracy of approximation of the given function by the modulus of the diagram one may use currents that oscillates much.

Keywords: antenna synthesis, nonlinear problem, modulus of the diagram, closed form solution.

1 Introduction

There is a large literature (e.g. [1], [2], [4]-[8], [13]) on finding $j(x)$ given the diagram

$$f(k) = \int_{-l}^{l} e^{ikx} j(x)dx \ := Aj, -k_0 \le k \le k_0. \tag{1}$$

Here $k_0 > 0$ and $l > 0$ are fixed numbers. The diagram $f(k)$ is an entire function of k, $|f(k)| \le ce^{l|k|}$, $k \in \mathbb{C}$. By $c > 0$ we denote various estimation constants. Equation (1) for j is an integral equation with compact operator $A : L^2(-l,l) \to L^2(-k_0,k_0)$. The operator A is injective, its range $R(A) := \{f \ : \ Aj, \ j \in L^2(-l,l)\}$ is not closed. Thus, if $\| f_\delta - f \|_{L^2(-k_0,k_0)} < \delta$, then f_δ may be not in $R(A)$.

In this paper the following problem is discussed:

Problem 1. Given $h(k) \ge 0$, $h(k) \in L^2(-k_0,k_0)$, *and* $\delta > 0$, *find* $j_\delta(x) \in L^2(-l,l)$ *such that*

$\| h(k) - | f_\delta(k) | \|_{L^2(-k_0,k_0)} < \delta$, *where* $f_\delta(k) = \int_{-l}^{l} e^{ikx} j_\delta(x)dx$.

This is a nonlinear problem. There was no closed form solution to this problem. The problem has been discussed in [1], where the approach was based on a numerical solution of a corresponding nonlinear minimization problem. Some theoretical aspects of antenna synthesis problem were discussed in [4]-[11], [13].

In Section 2 our main result, *Theorem 1*, is formulated. In Section 3 *Theorem 1* is proved. In Section 4 the numerical results, based on *Theorem 1*, are presented.

2 Closed Form Solution

Our approach is quite different from the one in [1], which reduces *Problem 1* to a *linear problem that is solved in closed form.* Our result is formulated in Theorem 1.

Denote

$$f = Fj := \int_{-l}^{l} e^{ikx} j(x)dx,\ \ j \in L^2(-l,l),\ \ j = 0,\ \ |x| > l, \tag{2}$$

and

$$F^{-1}f = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} f(k)dk. \tag{3}$$

Theorem 1. *A solution to Problem 1 is*

$$j_\delta(x) = 2\pi(F^{-1}G_n(\delta)\cdot(F^{-1}h), \tag{4}$$

where

$$G_n(k) = \left(\frac{1}{2l}\int_{-l}^{l} e^{\frac{ikx}{2n+p}}\right)^{2n+p}\left(\frac{n}{\pi k_1^2}\right)^{1/2}\left(1-\frac{k^2}{k_1^2}\right)^n,\quad n = n(\delta), \tag{5}$$

$k_1 > 2k_0$, $p > \frac{1}{2}$ *is fixed, and* $n = n(\delta)$ *is chosen so that*

$$\|\int_{-k_0}^{k_0} G_n(k-s)h(s)ds - h(k)\|_{L^2(-k_0,k_0)} < \delta\ . \tag{6}$$

Theorem 1 is proved in Section 3.

3 Proof of Theorem 1

We formulate and prove below several Lemmas from which the conclusion of Theorem 1 follows immediately.

Lemma 1. *The set*

$$\left\{f(k) = \int_{-l}^{l} e^{ikx} j(x)dx\right\}_{\forall j\in L^2(-l,l)}$$ *is dense in* $L^2(-k_0,k_0)$ *for any fixed* $k_0 > 0$.

Proof. If Lemma 1 is wrong, then there exists an element $h \in L^2(-k_0,k_0)$ such that

$$\int_{-k_0}^{k_0} h(k)\left(\int_{-l}^{l} e^{ikx} j(x)dx\right)dk = 0\quad \forall j \in L^2(-l,l). \tag{7}$$

This implies

$$\int_{-l}^{l} j(x)\left(\int_{-k_0}^{k_0} e^{ikx} h(k)dk\right)dx = 0\quad \forall j \in L^2(-l,l). \tag{8}$$

If a function is orthogonal to every $j \in L^2(-l,l)$, then this function vanishes identically:

$$\int_{-k_0}^{k_0} e^{ikx} h(k)dk = 0,\qquad -l \le x \le l\quad \forall j \in L^2(-l,l). \tag{9}$$

Since the integral in (9) is an entire function of x vanishing on an interval $x \in (-l,l)$, this function must vanish identically. In particular, it vanishes for $-\infty < x < \infty$. Therefore $h(k) = 0$ by the injectivity of the Fourier transform.□

The proof of the following lemma is based on the results from [9], [10], see also [11].

Lemma 2. *The function*

$$G_n(k) := \left(\frac{1}{2l}\int_{-l}^{l} e^{\frac{ikx}{2n+p}}\right)^{2n+p} \left(\frac{n}{\pi k_1^2}\right)^{1/2} \left(1-\frac{k^2}{k_1^2}\right)^n$$

is an entire function of exponential type l *, which belongs to* $L^2(-\infty,\infty)$ *if* $p > \frac{1}{2}$.

Proof. Clearly, the function $g_n(k) = \left(\frac{1}{2l}\int_{-l}^{l} e^{\frac{ikx}{2n+p}}dx\right)^{2n+p}$ is an entire function of exponential type l, $|g_n(k)| < ce^{l|k|}$. If $|k| \to \infty$ then

$$g(k) = \left(\frac{\sin\frac{kl}{2n+p}}{\frac{kl}{2n+p}}\right)^{2n+p} = O(\frac{1}{k^{2n+p}})$$

and

$$\delta_n(k) = \sqrt{\frac{n}{\pi k_1^2}}(1-\frac{k^2}{k_1^2})^n = O(k^{2n}).$$

Therefore the smooth function $G_n(k) = O(\frac{1}{|k|^p})$ as $|k| \to \infty$, so $G_n(k) \in L^2(-\infty,\infty)$ if $p > \frac{1}{2}$. □

Lemma 3. *The function* $\delta_n(k)$ *is a delta-sequence:*

$$\lim_{n\to\infty}\sqrt{\frac{n}{\pi k_1^2}}\int_{-\varepsilon}^{\varepsilon}(1-\frac{k^2}{k_1^2})^n dk = 1 \tag{10}$$

for any $\varepsilon \in (0,k_0)$, $k_1 > k_0$ *and*

$$\lim_{n\to\infty}\sqrt{\frac{n}{\pi k_1^2}}\int_{a}^{b}(1-\frac{k^2}{k_1^2})^n dk = 0 \tag{11}$$

if $0 \notin (a,b)$, *that is, if* $0 < a < b < k_0$ *or* $-k_0 < a < b < 0$.

Proof. The proof is easy and is left to the reader.

Lemma 4. *If* $h \in C^1(-k_0,k_0)$ *and* $\max_{-k_0\le k\le k_0} |h'| = M_1$, *then*

$$\int_{-k_0}^{k_0}\delta_n(k-s)h(s)ds = h(k) + O(\frac{1}{\sqrt{n}}), n\to\infty. \tag{12}$$

Proof. One has

$$\int_{-k_0}^{k_0}\delta_n(k-s)h(s)ds = h(k)\int_{-k_0}^{k_0}\delta_n(k-s)h(s)ds +$$
$$\int_{-k_0}^{k_0}\delta_n(k-s)[h(s)-h(k)]ds = h(k) + O(e^{-\gamma n}) + J, \gamma = \mathtt{const} > 0,$$

where

$$J \; := \int_{-k_0}^{k_0} \delta_n(k-s)[h(s)-h(k)]ds.$$

So, using the Lagrange formula with $t \in (s,k)$, one gets

$$J = |\int_{-k_0}^{k_0} \delta_n(k-s)h'(t)(s-k)ds| \le M_1 \int_{-k_0}^{k_0} \delta_n(k-s) \,|\, s-k \,|\, ds \le$$
$$M_1 \left(\frac{n}{\pi k_1^2}\right)^{1/2} \int_{-\frac{k_0+k}{k_1}}^{\frac{k_0-k}{k_1}} (1-u^2)^n \,|\, u \,|\, du \le 2k_1 M_1 \sqrt{\frac{n}{\pi}} \int_0^{\frac{2k_0}{k_1}} (1-u^2)^n u du = \tag{13}$$
$$k_1 M_1 \sqrt{\frac{n}{\pi}} \int_0^{4k_0^2/k_1^2} (1-v)^n dv \le k_1 M_1 \sqrt{\frac{n}{\pi}} \frac{1}{n+1} \le \frac{k_1 M_1}{\sqrt{\pi}} \frac{1}{\sqrt{n+1}}.$$

Here the following estimate is used:

$$\int_{-k_0}^{k_0} \delta_n(k-s)ds = 1 + O(e^{-\gamma n}), \quad \gamma = \frac{k_0-k}{k_1} \le \frac{2k_0}{k_1}. \tag{14}$$

Lemma 4 is proved.□

Define

$$f_n(k) := \int_{-k_0}^{k_0} G_n(k-s)h(s)ds. \tag{15}$$

This function is entire, of exponential type l, the same as the exponential type of G_n, and $f_n \in L^2(-\infty,\infty)$ if $p > \frac{1}{2}$, since $G_n(k)$ has this property. Therefore by the Paley-Wiener theorem [12], it follows that

$$F^{-1} f_n \; := j_n(x) \in L^2(-l,l),$$

where $j_n(x) = 0$ for $|x| > l, \; x \in \mathrm{R}$.

Lemma 5. *If* $n = n(\delta)$ *is sufficiently large, then*

$$|| f_n(k) - h(k) ||_{L^2(-k_0,k_0)} < \delta. \tag{16}$$

Proof. The conclusion of this lemma follows from two facts:

1) The function $g_n(k) \underset{n\to\infty}{\to} 1$ uniformly on the interval $|k| < k_1$;

2) $\delta_n(k)$ is a delta-sequence.

The second fact is proved in Lemma 3.

Let us prove the first fact. Since $\int_{-l}^{l} xdx = 0$, one has (cf. Lemma 2):

$$g_n(k) = \left(\int_{-l}^{l} e^{\frac{ikx}{2n+p}} dx\right)^{2n+p} = \left(1 + \frac{1}{2l}\int_{-l}^{l} \frac{1}{2}\left(\frac{ikx}{2n+p}\right)^2 dx + O\left(\frac{1}{n^3}\right)\right)^{2n+p}, \tag{17}$$
$$n \to \infty, \; |k| < k_1.$$

Thus

$$g_n(k) = 1 + O(\frac{1}{n}),\ n \to \infty,\ |k| < k_1 . \tag{18}$$

Relation (18) holds for any finite k_1.□

Lemma 6. *One has*

$$\| |f_n(k)| - h(k) \|_{L^2(-k_0,k_0)} < \delta . \tag{19}$$

Proof. Inequality (19) follows from (16), inequality $h(k) \ge 0$, and an elementary inequality

$$||f_n(k)| - h(k)| \le |f_n(k) - h(k)| .\ \square$$

Lemma 7. *One has*

$$j_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_n(k) e^{-ikx} dk = 2\pi F^{-1}G_n \cdot F^{-1}h, \tag{20}$$

where $F^{-1}h$ *is defined in (3).*

Proof. One has

$$\begin{aligned} &j_n(x) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} dk \int_{-k_0}^{k_0} G_n(k-s)h(s)ds = \\ &\int_{-k_0}^{k_0} dsh(s)e^{-isx}\, \tfrac{1}{2\pi}\int_{-\infty}^{\infty} G_n(k-s)e^{-(k-s)x} dk = \pi F^{-1}G_n \cdot F^{-1}h. \end{aligned} \tag{21}$$

Lemma 7 is proved.□

The conclusion of Theorem 1 follows from Lemmas 1-7 immediately.□

Remark 1. By estimate (16), formula (15) and Lemma 4 one has $\delta = O\left(\frac{1}{\sqrt{n}}\right)$, so $n = n(\delta) = O\left(\frac{1}{\delta^2}\right)$ as $\delta \to 0$.

Remark 2. By Parseval's equality one has

$$\begin{aligned} &\| j_n \|_1^2 := \int_{-l}^{l}(| j_n |^2 + | j_n'^2)dx = \tfrac{1}{2\pi}\int_{-\infty}^{\infty}(1+k^2)\,| f_n(k)|^2\, dk = \\ &\tfrac{1}{2\pi}\int_{-\infty}^{\infty}(1+k^2)\left|\int_{-k_0}^{k_0} G_n(k-s)h(s)ds\right|^2 dk. \end{aligned} \tag{22}$$

Thus, one can check the norm $\| j_n \|_1$ if h and G_n are given.

If $p > \frac{1}{2}$, then $j_n \in L^2(-l,l)$. If $p > 1$, then $\| j_n \|_1 < \infty$. By changing n and p one changes $\| j_n \|_1$.

Remark 3. Our method is valid if the linear segment $(-l,l)$ is replaced by a multidimensional bounded domain D. In this case the origin has to be chosen at the gravity center of D, that is, at the point such that $\int_D x dx = 0$. In this case formula (18) and Lemma 2 remain valid, the function $G_n(k)$ is

$$G_n(k) = \left(\frac{1}{|D|}\int_D e^{\frac{ik\cdot x}{2n+p}} dx\right)^{2n+p} \cdot c_{n,N}\left(1 - \frac{k\cdot k}{k_1^2}\right)^n, \tag{23}$$

where $k \cdot k$ is the dot product of vectors, $|D|$ is measure (volume) of D, N is the dimension of the space, and $c_{n,N}$ is the normalizing constant:

$$\lim_{n\to\infty} c_{n,N} \int\limits_{\substack{|k|<k_0 \\ 2k_0<k_1}} \left(1 - \frac{k \cdot k}{k_1^2}\right)^n dk = 1. \tag{24}$$

If D is smooth and strictly convex then the Fourier transform of the characteristic function of D is $O(\frac{1}{k})$. Therefore $G_n(k) = O(\frac{1}{k^p}) \in L^2(\mathrm{R}^N)$ if $p > \frac{N}{2}$. There is a large literature on the rate of decay of the Fourier transform of the characteristic function of a bounded domain D in R^N [3], [11].

4 Numerical Modeling

The numerical results related to investigation of the role of the number n on the quality of approximation of the desired diagram $h(k)$ are shown in Table 1. The parameters of the problem are the following: $l = 2.0$, $k_0 = 3.0$, $k_1 = 2k_0 + 1.5$, $p = 1.0$. The errors of the estimate (6) are given in the second column of Table 1. The mean-square deviation (MSD) $\|\, h(k) - |f(k)|\,\|$, obtained in the process of solving the nonlinear synthesis problem by approach in [1], is presented in the third column, and the square of the norm $\|j\|^2$ is given in the last column. One can see that the value of n influences strongly the accuracy of the approximation of the desired diagram. In order to get the error δ of the approximation which is less than 10^{-3} it is sufficient to choose n around 4000.

The quality of the approximation of the desired diagram $h(k)$ by $|f(k)|$ for small n is shown in Fig. 1. The given function $h(k)$ is plotted by thick solid line. The moduli $|f(k)|$ for $n = 10, 20, 50, 100$ are shown with the thin lines, the respective currents $j(x)$ are shown in Fig. 2. So, the error of estimate (6) for $n = 100$ is equal to 0.0295. The error of the approximation decreases if n increases, and its minimal value in our computations is equal to 0.7146×10^{-4} and is attained at $n = 4000$.

The values of the MSD, presented in the third column, are of the same order as the errors of estimate (6). For the given diagram $h(k)$ the increase of the accuracy of the approximation does not force the growth of the norm $\|j\|$ of the current which would cause practical difficulties (column 4 in Table 1).

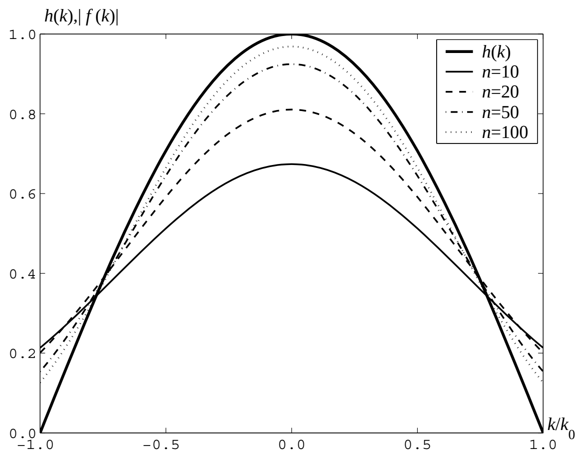

Fig. 1. The given $h(k) = \cos(\pi k / 2k_0)$ and the obtained $f(k)$ diagrams.

Table 1. Quality of the approximation of $h(x) = \cos(\pi k / 2k_0)$ for various n.

n	Est. (6)	MSD	$\|\| j(x) \|\|^2$
10	0.2784	0.2951	0.5060
20	0.1463	0.1495	0.5858
50	0.0591	0.0587	0.6432
100	0.0295	0.0287	0.6653
200	0.0147	0.0135	0.6772
500	0.0059	0.0044	0.6847
1000	0.0029	0.0014	0.6873
4000	0.7146×10^{-4}	0.8135×10^{-4}	0.6892

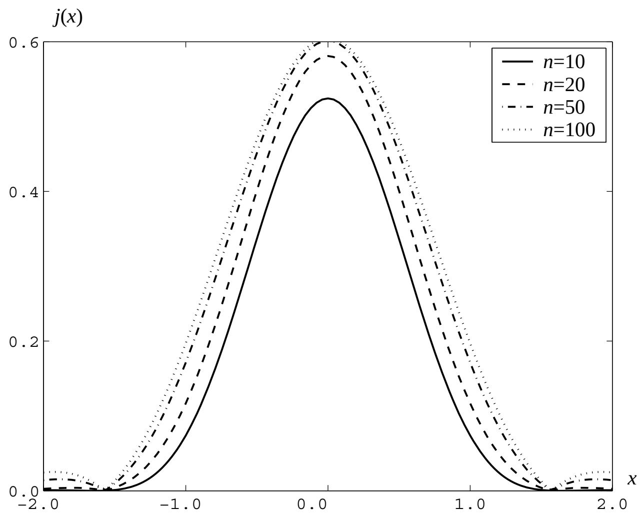

Fig. 2. The currents $j(x)$ for various n, $h(k) = \cos(\pi k / 2k_0)$.

The quality of approximation to the given diagram $h(k) = 1$ is shown in Table 2. For this $h(k)$ the error of the approximation is larger than the error for $h(k) = \cos(\pi k / 2k_0)$ at the same values of n. The value of δ in estimate (6) at $n = 4000$ is two orders greater than that for the diagram $h(k) = \cos(\pi k / 2k_0)$. Although the error of estimate (6) and the MSD are small, but the difference of the shapes of $h(k)$ and $f(k)$ is visible. In the four last rows of Table 2 the results are presented for $k_0 = 6.0$ and $k_0 = 9.0$. The error of estimate (6) is almost the same as for $k_0 = 3.0$, but the value of the MSD is lower. This means an improvement of the approximation to the given diagram by the shape (compare the dash dot and dot curves in Fig. 3). The mean-square deviation at $n = 1000$ and $n = 4000$ for $k_0 = 9.0$ is almost two times less than for $k_0 = 3.0$.

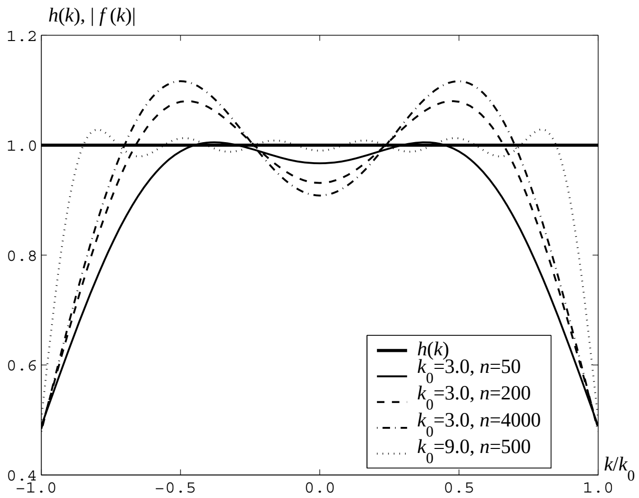

Fig. 3. The given $h(k) = 1$ and the obtained $| f(k) |$ diagrams.

The corresponding distributions of the current $j(x)$ are shown in Fig. 4. For larger k_0 the norm of the current grows. This agrees with the numerical results in [1], namely the better approximation of the given diagram leads to the larger norm of the current.

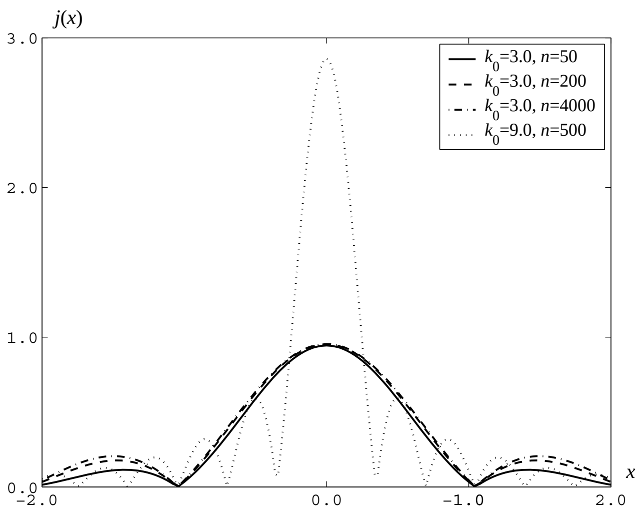

Fig. 4. The currents $j(x)$ for several parameters k_0, $h(k) = 1$.

The number of n, which is sufficient for obtaining the desired error δ, is shown in Table 3 for $l = 2.0$, $k_0 = 6.0$, $k_1 = 2k_0 + 1.5$, and $p = 1.0$. The results are presented for the given diagrams $h(k) = (\cos(\pi k / 2k_0)^{2q}$, with $q = 1, 4, 8, 16, 32$. To obtain a higher accuracy of approximation of h it is necessary to increase the number n for all q. The quantity n for the prescribed δ varies for different q.

Table 2. Quality of the approximation of $h(x)=1$ for various n.

	n	Est. (6)	MSD	$\|\| j(x) \|\|^2$
$k_0 = 3.0$				
	10	0.5592	0.5885	0.7287
	20	0.3654	0.3712	0.8314
	50	0.2161	0.2219	0.8967
	100	0.1487	0.1650	0.9218
	200	0.1034	0.1338	0.9361
	500	0.0645	0.1141	0.9456
	1000	0.0433	0.1073	0.9458
	4000	0.0225	0.1021	0.9515
$k_0 = 6.0$				
	500	0.0584	0.0715	1.3506
	4000	0.0203	0.0537	1.3624
$k_0 = 9.0$				
	500	0.0565	0.0607	1.6575
	4000	0.0195	0.0380	1.6749

The power q in the function $(\cos(\pi k / 2k_0)^{2q}$ corresponds to the diagrams with different widths at the level $h(k)=0.5$. The value of $| h(k) - | f(k) ||_{h(k)=0.5}$ is minimal at $q=16$. Also, this $f(k)$ has the smallest side lobes outside the interval $| k | \leq k_0$ in comparison with the values of the side lobes at other values of q. Such diagram is called optimal [2], and it can be created more easily in comparison with other diagrams. This leads to the minimal value of n which is necessary to obtain the desired error δ.

5 Conclusion

The closed form solution of the linear antenna synthesis problem by the modulus of the diagram is given. The initial nonlinear problem is reduced to a linear one. The solution of this linear problem is presented in closed form, see formula (4).

The numerical results, presented for several given amplitude diagram, demonstrate the high accuracy of the proposed method. The approach can be used also for solving the multidimensional antenna synthesis problems.

Table 3. Number of n necessary to attain the given value of δ for various $h(k)$.

$h(k)$	$\delta = 0.1$	$\delta = 0.01$	$\delta = 0.001$
$h(k) = (\cos(\pi k / 2k_0)^2$	$n = 24$	$n = 86$	$n = 320$
$h(k) = (\cos(\pi k / 2k_0))^8$	$n = 15$	$n = 50$	$n = 240$
$h(k) = (\cos(\pi k / 2k_0))^{16}$	$n = 12$	$n = 44$	$n = 165$
$h(k) = (\cos(\pi k / 2k_0))^{32}$	$n = 9$	$n = 46$	$n = 150$
$h(k) = (\cos(\pi k / 2k_0))^{64}$	$n = 13$	$n = 55$	$n = 182$

References

1. Andriychuk, M., Voitovich, N., Savenko, P., Tkachuk, V.: Antenna synthesis by amplitude radiation pattern. Nauk. Dumka, Kiev (1993).
2. Balanis, C. A.: Antenna theory: analysis and design, (3rd ed.). Wiley-Interscience, Hoboken, New Jersey (2005).
3. Iosevich, A., Liflyand, E.: Decay of the Fourier transform: analytic and geometric aspects. Birkhäuser, Basel (2014).
4. Minkovich, B., Yakovlev, V.: Theory of antenna synthesis. Sov. Radio, Moscow (1969).
5. Ramm, A. G.: Optimal solution of the problem of linear antenna synthesis. Soviet Physics – Doklady. 13, No 6, 546-548 (1968).
6. Ramm, A. G.: On antenna synthesis theory. Collection "Antennas". Izd. Svjaz, Moscow. No 5, 35-46 (1969).
7. Ramm, A. G.: Some nonlinear problems in the theory of antenna synthesis. Soviet Physics – Doklady. 14, No 6, 532-535 (1989).
8. Ramm, A. G.: Optimal solution of the linear antenna synthesis problem. Radiofisika. 12, 1842-1848 (1969).
9. Ramm, A. G.: Approximation by entire functions, Mathematics. Izv. Vusov. 10, 72-76 (1978).
10. Ramm, A. G.: Signal estimation from incomplete data. J. Math. Anal. Appl. 125, 267-271 (1987).
11. Ramm, A. G., Katsevich, A. I.: The Radon transform and local tomography. CRC Press, Boca Raton (1996).
12. Rudin, W.: Real and complex analysis. McGraw Hill, New York (1988).
13. Stutzman, W., Thiele, G.: Antenna theory and design. Wiley, London (2013).

