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Cosmological Models with Variable G and Λ in Kaluza-Klein Space-time

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Abstract. We have investigated new cosmological models in five-dimensional Kaluza-Klein space-time with a variable gravitational constant 'G' and cosmological constant 'Λ'. The Einstein's field equations have been solved for Kaluza-Klein space-time in the presence of a perfect fluid with time dependent G and Λ. We find a variety of solutions in which G increases and Λ decreases with time t. The extra dimension becomes insignificant as $t \to \infty$ and we are left with the real four dimensional universe.

Keywords: Five-dimensional, Kaluza-Klein space-time, variable G & Λ.

1 Introduction

The High-z supernova search team (Tonry et al, 2003 [1]) confirms the results of High Red Shift Type Ia Supernova and Supernova Cosmological Project team (Garnavich et al, 1998 [2], Perlmutter et al, 1997 [3], 1998 [4], 1999 [5], Riess et al, 1998 [6], Schmidt et al, 1998 [7]) that a positive cosmological constant of order ($Gh/c^3 \approx 10^{-123}$) may dominate the total energy density in the universe (Sahni, 2000 [8]). These observational analyses also indicate that the supernova luminosity distances imply an accelerating universe with total negative pressure for the universe.

The cosmological models formulated for higher dimensional space-time play a vital role in many aspects of early stage of cosmological problems, which are one of the frontier areas of research to unify gravity with other forces in nature. The study of higher dimensional space-time provides an idea that our universe was much smaller at an earlier stage of evolution than observed today and extra dimensions were contracted to a very small dimension. The detection of extra dimensions in current experiments is beyond those four dimensions observed so far.

The possibility of extra dimensions in the space-time has attracted various researchers to the field of cosmology. The field of cosmology has been highly enriched by the Kaluza [9] and Klein [10] theory, in which they have shown that gravitation and electromagnetism could be unified in a single geometrical structure. Chodas and Detweiler (1980 [11]) obtained higher dimensional cosmological model in which extra dimension contracts and implies that this contraction of extra dimension is due to consequences of cosmological evolution. The extra dimensions produce massive amount of entropy during the contraction process, as shown by Guth (1981 [12]), and Alvarez and Gavela (1983 [13]). This provides an alternative solution to the flatness and horizon problems as compared to the usual inflationary scenario. In the previous years, a number of authors (Freund, 1982 [14], Appelquist and Codos, 1983 [15], Randjbar-Daemi et al, 1984 [16], Rahaman et al, 2002 [17], Singh et al, 2004 [19], Khandetar et al, 2006 [20], Yilmaz and Yavuz, 2006 [21], Mohanty et al, 2006 [22], 2007 [23], Pradhan et al, 2007 [24]) obtained the solutions of Einstein's field equations for higher dimensional space-times containing a variety of matter fields. In their analysis, some authors have shown that there is an expansion of four dimensional space-times while the fifth dimension contracts or remains constant.

In 1937, Dirac [25] proposed the idea of a variable gravitational constant G in the framework of general relativity. The modifications linking the variation of G with that of Λ have been proposed within the framework of general relativity by Lau (1985 [26]). This alteration allow us to use Einstein's field equations formally unchanged since a variation in Λ is accompanied by a variation of G. The above present approach is non-covariant and Einstein's field equations cannot be obtained from a Hamilitonian. This approach solves many cosmological problems, viz., the cosmological constant problem, the initial singularity problem and inflationary universe scenario. As mentioned by Kalligas et al (1992 [32]), this approach may be the limit of more viable, fully covariant theory, such as a five-dimensional Kaluza-Klein theory.

Cosmological models in higher-dimensional space-time with variable gravitational constant G and cosmological constant Λ have been studied by a limited number of authors. Chakraborty and Ghosh [27], Rahaman and Bera [28], and Rahaman et al, (2006 [18]) obtained cosmological models in Kaluza-Klein Space-time for the context of different geometries. Baysal and Yilmaz, (2007 [30]) investigated cosmological models in five-dimensional Kaluza-Klein space-time with variable G and Λ.

The goal of this article is to investigate physically sound cosmological models in five-dimensional Kaluza-Klein space-times with variable G and Λ. In this paper, we have solved the Einstein's field equations for five-dimensional Kaluza-Klein space-time in the presence of perfect fluid with time dependent G and Λ.

2 Einstein's Field Equations for Kaluza-Klein Space-times.

The Five-dimensional Kaluza-Klein space-time is described by [10]

$$ds^2 = -dt^2 + X^2(t)\left(dx^2 + dy^2 + dz^2\right) + A^2(t)d\psi^2 \tag{1}$$

The Einstein's field equations with time-dependent gravitational and cosmological 'constants' are given by

$$R_{ij} - \frac{1}{2}g_{ij}R = -8\pi G(t)T_{ij} - \Lambda(t)g_{ij} \tag{2}$$

Energy momentum tensor for a one-fluid source is

$$T_{ij} = (p + \rho)u_i u_j - pg_{ij} \tag{3}$$

with

$$g^{ij}u_i u_j = 1 \tag{4}$$

ρ is the energy density and p is the isotropic pressure of the fluid. u_i is the five-velocity of the time-like vector satisfying the equation (4). The off diagonal equations of (2) together with energy conditions $p + \rho \geq 0, \rho \geq 0$ imply that:

$$u_i = (0,0,0,0,1) \tag{5}$$

Using equations (2), (3), and (5), surviving field equations (1) are:

$$3\frac{\dot{X}\dot{A}}{XA} + 3\frac{\dot{X}^2}{X^2} = 8\pi G\rho + \Lambda \tag{6}$$

$$3\frac{\ddot{X}}{X} + 3\frac{\dot{X}^2}{X^2} = -8\pi Gp + \Lambda \tag{7}$$

$$2\frac{\ddot{X}}{X} + \frac{\ddot{A}}{A} + \frac{\dot{X}^2}{X^2} + 2\frac{\dot{X}\dot{A}}{XA} = -8\pi Gp + \Lambda \tag{8}$$

The usual conservation law for Einstein's field equation $T^{ij}_{;j} = 0$ (; i.e. semicolon denotes covariant divergence) implies:

$$\dot{\rho} + \left(3\frac{\dot{X}}{X} + \frac{\dot{A}}{A}\right)\left(\rho + p\right) = 0 \tag{9}$$

In Einstein's theory, the principle of equivalence must require that G and Λ not enter the equation of motion of particle and photons, i.e. only g_{ij} must enter them. The vanishing of the covariant divergence of the Einstein tensor in equation (2) and together with equation (9), we obtain

$$\dot{\Lambda} = -8\pi\dot{G}\rho \tag{10}$$

Here, an overdot denotes a derivative with respect to cosmic time t. These are four equations (6 -9 or 10) in six unknowns. Thus to get a solution we require two additional relations. These relations may be taken to involve field variables as well as physical variables. In the following sections we shall explore the possibility of finding physically meaningful solutions of the field equations subject to specified geometrical and physical conditions.

3 Solutions of Field Equations.

One of the relation is the equation of state

$$p = w\rho \tag{11}$$

From equations (7) and (8) an equation connecting X and A is obtained:

$$\frac{\ddot{X}}{X} - \frac{\ddot{A}}{A} + 2\frac{\dot{X}^2}{X^2} - 2\frac{\dot{X}\dot{A}}{XA} = 0 \tag{12}$$

Another relation is the power law from one of the field parameters, that is

$$X = t^n \tag{13}$$

where w and n are constant. Substituting the equation (13) to equation (12) and integrating, we obtain

$$A = \alpha t^n + \beta t^{(1-3n)} \tag{14}$$

Using the equations (13), (14) and (11), the resulting solution can be expressed as follows:

$$\rho = \delta\left[\alpha t^{4n} + \beta t\right]^{-(1+\omega)} \tag{15}$$

$$\frac{(1+\omega)}{3}\Lambda = \frac{\left[(2\omega n^2 + 2n^2 - n)\alpha t^{(4n-1)} + (2n^2 - 2\omega n^2 + \omega n - n)\beta\right]}{t^2\left[\alpha t^{(4n-1)} + \beta\right]} \tag{16}$$

$$\frac{8\pi\delta(1+\omega)}{3}G = \frac{\left(\alpha t^{(4n-1)} + \beta\right)^{\omega}\left(n\alpha t^{(4n-1)} + n(2-4n)\beta\right)}{t^{(1-\omega)}} \tag{17}$$

The cosmological parameters are given by:

$$\theta = \frac{\left[4n\alpha t^{(4n-1)} + \beta\right]}{t\left[\alpha t^{(4n-1)} + \beta\right]} \tag{18}$$

$$\sigma^2 = \frac{3}{8}\frac{(4n-1)^2\beta^2}{t^2\left[\alpha t^{(4n-1)} + \beta\right]^2} \tag{19}$$

$$q = \frac{4n(3-4n)\alpha^2 t^{2(4n-1)} + 4n(7-12n)\alpha\beta t^{(4n-1)} + 2\beta^2}{\left[4n\alpha t^{(4n-1)} + \beta\right]^2} \tag{20}$$

where α and β are non-negative integration constants. The solution has one or two distinct singularities viz. $t = 0$ and $t = \frac{\alpha}{\beta}$ depending upon the sign of the constants of integration. We come across six types of situations depending upon the parameters w, α and β:

Case I: $\alpha = 0$. In this case, $X = t^n$, $A = \beta t^{(1-3n)}$, $\rho = \delta\left(\beta t\right)^{-(1+\omega)}$, $\Lambda = \frac{3n(2n-1)(1-\omega)}{(1+\omega)t^2}$, $8\pi\delta G = \frac{6n(1-2n)\beta^{(1+\omega)}}{(1+\omega)t^{(1-\omega)}}$, $\theta = \frac{1}{t}$, $\sigma = \sqrt{\frac{3}{8}}\frac{(4n-1)}{t}$ and $q = 2$. Putting $n = \frac{1}{(m+3)}$, we obtain the solution given by Baysal and Yilmaz [30].

Case II: $\beta = 0$. In this case, $X = t^n$, $A = \alpha t^n$, $\rho = \delta\left(\alpha t^{4n}\right)^{-(1+\omega)}$, $\Lambda = \frac{3(2\omega n^2 + 2n^2 - n)}{(1+\omega)t^2}$, $8\pi\delta G = \frac{3n(\alpha t^{(4n-1)})^{(1+\omega)}}{(1+\omega)t^{(1-\omega)}}$, $\theta = \frac{4n}{t}$, $\sigma = 0$ and $q = \frac{3-4n}{4n}$, $w \neq -1$. For $n = \frac{1}{2(1+\omega)}$, Λ becomes zero and $G = \frac{3n\,\alpha^{(1+\omega)}}{16\pi\delta(1+\omega)^2} = constant$, which corresponds to a radiation dominated universe for $n = \frac{3}{8}$ and matter dominated universe for $n = \frac{1}{2}$ with Einstein gravity. Λ is decreases with time t as time increases and Λ is non-negative, for $n < 0 and\, n > \frac{1}{2(1+w)}$; whereas for $0 < n < \frac{1}{2(1+w)}$, Λ is negative. The parameter q becomes insignificant for $n = 3/4$ and negative for $n < 0$ *and* $n > 3/4$. For $-1 < w \leq 1$, it gives the range for the parameter (n) $1/4 \leq n < \infty$. Here for a particular choice of

$n > 3/4$, we can find the q and w negative, and Λ small positive constant, which can support the observations by supernova cosmology project.

Case III: $\omega = 0$. In this case, $X = t^n$, $A = \alpha t^n + \beta t^{(1-3n)}$, $\rho = \frac{\delta}{\left[\alpha t^{4n} + \beta t\right]}$, $\Lambda = \frac{3n(2n-1)}{t^2}$ and $G = \frac{3n\left[\alpha t^{(4n-1)} + 2(1-2n)\beta\right]}{8\pi\delta t}$. For $n = 0$, Λ and G become insignificant and $\rho \propto \frac{1}{t}$. For $n = \frac{1}{2}$, $\rho \propto \frac{1}{t}$ and gravitational constant G is constant and cosmological constant is absent. This case is similar to the cosmological solutions obtained by Oli (2008 [33]) for $n = \frac{2}{3}$ (case II). Λ is decreasing function of t and for $n < 0$ and $n > \frac{1}{2}$, Λ is non-negative; whereas for $0 < n < \frac{1}{2}$, Λ is negative. For $n < \frac{1}{2}(n \neq 0)$, G<0 for all t and decreases monotonically as t increases. For $n > \frac{1}{2}$, G is non-negative for all t and becomes insignificant for $t = t_* = \left[\frac{2(2n-1)\beta}{\alpha}\right]^{\frac{1}{(4n-1)}}$ and increases monotonically afterwards.

Case IV: $\omega = 1$. In this case, $X = t^n$, $A = \alpha t^n + \beta t^{(1-3n)}$, $\rho = \frac{\delta}{t^2\left[\alpha t^{(4n-1)} + \beta\right]^2}$, $\Lambda = \frac{3n(4n-1)\alpha t^{(4n-1)}}{2t^2\left[\alpha t^{(4n-1)} + \beta\right]}$ and $G = \frac{3n\left[\alpha t^{(4n-1)} + \beta\right]\left[\alpha t^{(4n-1)} + 2(1-2n)\beta\right]}{16\pi\delta}$. For $n = 0$, Λ and G become insignificant and $\rho \propto \frac{1}{t^2}$. For $n = \frac{1}{4}$, $\rho \propto \frac{1}{t^2}$ and G is constant and cosmological constant is zero. Λ is decreasing function of time t. For $n < 0$ and $n > \frac{1}{4}$, Λ is non-negative; whereas for $0 < n < \frac{1}{4}$, Λ is negative. For $n < 0$, G<0 for all t and as $t \to \infty$, the gravitational constant become constant. For $0 < n \leq \frac{1}{2}(n \neq \frac{1}{4})$, G>0 for all t. In the range $0 < n < \frac{1}{4}$, the behavior of G is similar to the above; whereas**,** G is increasing monotonically as t increases. For $n > \frac{1}{2}$, G becomes insignificant for $t = t_* = \left[\frac{2(2n-1)\beta}{\alpha}\right]^{\frac{1}{(4n-1)}}$. For $0 < t < t_*$, G is negative. For $t > t_*$, G is non-negative and increases monotonically afterwards.

Case V: $\omega = -1$. In this case, $\rho = \delta$ is constant and Λ and G diverge. This corresponds to static universe with dark energy.

Case VI: $\omega = \frac{1}{3}$. In this case, $X = t^n$, $A = \alpha t^n + \beta t^{(1-3n)}$, $\rho = \frac{\delta}{\left[\alpha t^{4n} + \beta t\right]^{\frac{4}{3}}}$, $\Lambda = \frac{3n\left[(8n-3)\alpha t^{(4n-1)} + 2(2n-1)\beta\right]}{4t^2\left[\alpha t^{(4n-1)} + \beta\right]}$ and $G = \frac{9n\left[\alpha t^{(4n-1)} + \beta\right]^{\frac{1}{3}}\left[\alpha t^{(4n-1)} + 2(1-2n)\beta\right]}{32\pi\delta t^{\frac{2}{3}}}$. For $n = 0$, Λ and G become insignificant and $\rho \propto \frac{1}{t^{4/3}}$. For $n = \frac{1}{4}$, $\rho \propto \frac{1}{t^{4/3}}$ and G is non-negative and a decreasing function of time t; whereas cosmological constant is negative and decreases square of time t. Λ is decreasing function of time t and non-negative for $n < 0$ and $n \geq \frac{1}{2}$; whereas for $0 < n \leq \frac{1}{2}$, Λ is negative. In the range $\frac{3}{8} < n < \frac{1}{2}$, Λ becomes insignificant for $t = t_* = \left[\frac{2(2n-1)\beta}{(8n-3)\alpha}\right]^{\frac{1}{(4n-1)}}$ and increases monotonically afterwards. For $n \leq \frac{3}{8}(n \neq 0)$, G is decreasing function of time t; whereas for $\frac{3}{8} < n < \frac{1}{2}$, G increases monotonically for all t. For $n \geq \frac{1}{2}$, the behavior of G is similar to case-III given above.

4 Conclusion

In the previous sections we have presented five dimensional Kaluza-Klein cosmological models with a variable gravitational constant G and cosmological constant Λ and discussed six different cases for the particular value of w, α and β.

We obtain the class of cosmological models where gravitational constant G is increasing or decreasing with time t. The cosmological constant Λ decreases with time t in all cases, which is supported by results from recent supernovae Ia observations [8]. In case I, for the particular value of $n = \frac{1}{(m+3)}$, we retrieve the cosmological model obtained by Baysal and Yilmaz [30].

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