Measuring Risks in a Portfolio of Financial Assets using the Downside Risk Method

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Abstract Measuring risks in a portfolio of financial assets is a very high profile issue in financial mathematics research. In this article, we focus on the search for an efficient portfolio and a smooth efficient frontier using the Downside Risk method measured by the Semi-Variance to have a portfolio with minimal variance. More specifically, and considering a set of financial securities, we compare the Markowitz Mean-Variance method and the Downside Risk method. We find that Downside Risk is a better measure of risk than Mean-Variance and is therefore more suitable for building a portfolio of minimum variance.

Keywords: downside risk, average variance, efficient portfolio, efficient frontier, risk

1 Introduction

Risk measures has always played a very important role in corporate decision making. The development and implementation of financial markets, and the multi-study of financial products have stimulated researchers to develop models to help investors make the right decisions and implement effective strategies (risk measures), for a better fructification of their wealth in order to maximize their gains while minimizing the risk of losing. This need has given rise to a range of research work that has focused on portfolio management and the evolution of financial assets. The origin of the development of modern portfolio theory can be traced back to Markowitz (1952) [3,4] who determined the Mean-Variance model to measure the relationship between the return and the risk of portfolios. However, this model is valid only if returns are normally distributed or if investors have mean-variance preferences. However, several studies have shown that returns on financial assets are not normally distributed. Similarly, Ballestro (2005) [5,6], Estrada(2004) [7] have pointed out that variance is a dubious risk measure because it treats above-average and below-average returns in the same way, whereas investors associate risk with the target rate of return (benchmark). Thus, the mean-variance model becomes unable to adequately capture the characteristics of portfolio returns and investors' perceptions of risk. As a result, the use of the Downside Risk measure seems necessary in order to better present investor preferences and to take into account the asymmetric nature of returns. In this regard, Harlow (1991) developed a portfolio optimization model under Downside Risk, in which investors minimize only returns below the target rate of return while looking for desirable returns above the target. However, Harlow's model is not perfect, and it is with this in mind that Daboussi (2006), inspired by Estarda (2003) [8], determined a new portfolio optimization model for downside risk. Similar facts were also mentioned in Jimbo et al (2017) and Jimbo and Craven (2011).

The aim of this article is to present on the one hand the mathematical tools for measuring and comparing risk by developing respectively the theory of an optimal portfolio according to the Markowitz Mean-Variance model and the Downside Risk method. On the other hand, we will end our study by comparing the two models on optimal portfolios chosen in an efficient financial market.

2   Optimal Portfolio Concept by the Markowitz Method (1952)

2.1   Modern Portfolio Theory

The optimization of a portfolio or the optimal choice of a portfolio of financial assets is a very high profile issue in financial mathematics research. In this context, Markowitz was the first to introduce a model called "Mean-Variance" (1952) as a risk measurement for optimal portfolio choice. Indeed, the Markowitz model consists of minimizing the standard deviation for a given profitability or maximizing the portfolio's profitability for a given risk.

2.2   Characteristic of a Portfolio

Portfolio return   Let P_t be the price of a share at the end of period t, the variation of the price $(P_t - P_{t-1})$ designates the gain to which is possibly added the income d_t called dividend paid during period t.

The return on this share in the period t is defined as follows:

$$R_t = \frac{(P_t - P_{t-1}) + d_t}{P_{t-1}} \tag{1}$$

Let P be a portfolio of financial stocks $(A_1, ..., A_N)$ represented by a vector $x=(x_1, ..., x_N)$ where x_i denotes the proportion of the capital C invested in the stock a_i characterized by its uncertain return R_i $(i = 1, ..., N)$.

The profitability of a portfolio is equal to the weighted average of the securities or stocks that make it up. We have:

$$R_P = x_1 \times R_1 + ... + x_N \times R_N = \sum_{i=1}^{N} x_i R_i \tag{2}$$

Expected profitability of a portfolio   Under the above assumptions, the expected profitability of a portfolio is given by:

$$E[R_P] = E\left[\sum_{i=1}^{N} x_i R_i\right] = \sum_{i=1}^{N} E[x_i R_i] = \sum_{i=1}^{N} x_i E[R_i] \tag{3}$$

Consider a portfolio of N securities. For each security where assets A_j, we have T historical returns rated $R_j=[r_{j1}\ r_{j2}\ r_{j3}\ ....r_{ji}......r_{jT}]$. Generally, we have for all securities,

$$R = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1T} \\ r_{21} & r_{22} & \cdots & r_{2T} \\ \vdots & \vdots & \ddots & \vdots \\ r_{N1} & r_{N2} & \cdots & r_{NT} \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_N \end{pmatrix} \tag{4}$$

where R is the matrix of the returns of the N securities collected over T period. We can therefore determine the column vector of the mathematical expectations μ of dimension $N \times 1$.

$$\mu = E[R] = E\left[\begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_N \end{pmatrix}\right] = \begin{pmatrix} E[R_1] \\ E[R_2] \\ \vdots \\ E[R_N] \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{pmatrix} \tag{5}$$

Variance of portfolio return The variance covariance matrix Σ of dimension $N \times N$ is written as follows:

$$
\begin{aligned}
\Sigma = Var[R] &= \begin{pmatrix}
Var_[1, R_1] & Var_[R_1, R_2] & \cdots & Var_[R_1, R_N] \\
Var_[R_2, R_1] & Var_[R_2, R_2] & \cdots & Var_[R_2, R_N] \\
\vdots & \vdots & \ddots & \vdots \\
Var_[R_N, R_1] & Var_[R_N, R_2] & \cdots & Var_[R_N, R_N]
\end{pmatrix} \\
&= \begin{pmatrix}
Var_[R_1, R_1] & Cov[R_1, R_2] & \cdots & Cov_[R_1, R_N] \\
Cov_[R_2, R_1] & Var_[R_2, R_2] & \cdots & Cov_[R_2, R_N] \\
\vdots & \vdots & \ddots & \vdots \\
Cov_[R_N, R_1] & Cov_[R_N, R_2] & \cdots & Var_[R_N, R_N]
\end{pmatrix} \\
&= \begin{pmatrix}
\sigma_1^2 & \sigma_{1,2} & \cdots & \sigma_{1,N} \\
\sigma_{1,2} & \sigma_2^2 & \cdots & \sigma_{2,N} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{1;N} & \sigma_{N,2} & \cdots & \sigma_N^2
\end{pmatrix}
\end{aligned}
\tag{6}
$$

where we have on the diagonal the variances of each security. It should also be noted that our covariance matrix is symmetrical i.e. $\Sigma = \Sigma^T$. Thus, it is possible to build a portfolio $P(\omega)$ with:

$$
\omega = \begin{pmatrix}
\omega_1 \\
\omega_2 \\
\vdots \\
\omega_N
\end{pmatrix}
\tag{7}
$$

which represents the weight vector of each security or asset in the portfolio so the sum is equal to 1 in other words $\omega^T u = 1$ with:

$$
u = \begin{pmatrix}
1 \\
1 \\
\vdots \\
1
\end{pmatrix}
\tag{8}
$$

The expectation μ_P and the variance σ_P^2 of the portfolio can be expressed like this:

$$
\begin{cases}
\mu_P = \omega^T \mu \\
\Sigma_P = \omega^T \Sigma \omega
\end{cases}
\tag{9}
$$

Example 1. *In the case of a portfolio consisting of two assets, the variance is:*

$$
\begin{aligned}
Var[R_P] &= x_1^2 Var[R_1] + x_2^2 Var[R_2] + 2x_1 x2 Cov(R_1, R_2) \\
Var[R_P] &= x_1^2 Var[R_1] + x_2^2 Var[R_2] + 2x_1 x_2 \sigma_{R_1} \sigma_{R_2} Corr(R_1, R_2)
\end{aligned}
\tag{10}
$$

$$
\sigma_{R_P} = \sqrt{Var[R_P]}
\tag{11}
$$

Covariance and portfolio profitability correlation The variance of a multi-security portfolio is:

$$
\sigma^2 = \sum_i \sum_j x_i y_j Cov_{ij}
\tag{12}
$$

where Cov_{ij} is the covariance of securities i and j. It is used to measure the risk of a portfolio in the same lines as correlation.

Correlation measures the strength of the relationship between the returns of two securities and their propensity to move together. It ranges from +1 (the changes in returns are identical) to -1 (returns always move in opposite directions). When the correlation is zero, the changes in profitability are unrelated. Before measuring the risk in a portfolio of financial assets, we will first measure the risk of an asset because a portfolio is made up of several assets.

Measuring the risk of an asset The risk of an asset or security i is assessed by its contribution to portfolio risk. Two related measures can be given:

- **An absolute measurement** is the covariance (σ_{ip}) of the asset with the portfolio. σ_{ip} gives us the contribution of security i to the portfolio risk. It is the measurement of risk in the portfolio.
- **A relative measurement** is the Beta (β_{ip}) of the assets in the portfolio.

The covariance of an asset with the portfolio is the weighted average of the covariances of the assets with the portfolio.

$$\sigma_{ip} = \sum_j x_j \sigma_{ij} \tag{13}$$

In this case the variance of the portfolio is equal to the weighted average of the covariances of the assets with the portfolio.

$$\sigma_p^2 = \sum_i^N x_i \sigma_{ip} \tag{14}$$

As an illustration, let's look at the variance of a portfolio made up of two risky securities:

$$\sigma_p^2 = x_1^2 \sigma_1^2 + x_2^2 \sigma_2^2 + 2(x_1 x_2 \sigma_1 \sigma_2 \rho_{12}) \tag{15}$$

We can still write this expression in the following form:

$$\sigma_p^2 = x_1(x_1 \sigma_1^2 + x_2 \sigma_{12}) + x_2(x_1 \sigma_{12} + x_2 \sigma_2^2) \tag{16}$$

The terms in brackets are the covariance of each of the securities with the portfolio. For example, in the case of security 1, we have:

$$\begin{aligned} x_1 \sigma_1^2 + x_2 \sigma_{12} &= x_1 Cov(r_1, r_1) + x_2 Cov(r_1, r_2) \\ &= Cov(r_1, x_1 r_1) + Cov(r_1, x_2 r_2) \\ &= Cov(r_1, r_p) = \sigma_{1p} \end{aligned} \tag{17}$$

We obtain $\sigma_p^2 = x_1 \sigma_{1p} + x_2 \sigma_{2p}$. This expression is generalized in the case of N assets (we have already done it above).

The beta of security i relative to portfolio P is the ratio of the covariance of security i to the variance of the portfolio.

$$\beta_{ip} = \frac{\sigma_{ip}}{\sigma_p^2} \tag{18}$$

A beta greater than unity means that the risk of security j in portfolio P is above average. The average of the beta is equal to unity, i.e.:

$$\sum_{i=1}^N x_i \beta_{ip} = 1 \tag{19}$$

Determination of beta Beta is the slope of the following regression line.

$$r_j = \alpha + \beta r_p + \epsilon \tag{20}$$

2.3 Markowitz Method for the Determination of an Efficient Portfolio, Tangent Portfolio

Definition 1. *(Efficient portfolio)*
An efficient portfolio is any portfolio that offers the highest expected return for a given level of risk. Or conversely, any portfolio that exposes its holder to the lowest risk for a given level of return.

Definition 2. *(Efficient Frontier)*
The efficient frontier is the curve on which all efficient portfolios are represented.

From the notion of efficiency thus defined, we understand that there is only one and only one efficient portfolio $P(\omega^*)$ for an expected level of return μ_0. This is called the **tangent portfolios**.

Markowtiz explains and formalizes the fundamental dilemma of modern finance in the following way: *"To achieve low but certain profitability, or to accept risk in the hope of increasing that profitability. The expectation of profitability being higher than the risk is important."*

The first question an investor asks himself is obviously the following: Which efficient portfolio offers the most reliable level of risk? Answering this question means solving the following optimization program issued by Markowitz:

$$\begin{cases} Min\ \sigma_P^2(\omega) = \omega^T \Sigma \omega \\ \omega^T u = 1 \end{cases} \tag{21}$$

where

$$\begin{cases} MaxE(\mu_p) = Max\ \omega^T \mu \\ \sum_{i=1}^m \sum_{j=1}^m \sigma_{ij} = (\sigma_p^2)^* \\ \sum_{j=1}^m \omega_j = 1 \end{cases} \tag{22}$$

To solve this program, we will use the Lagrange multiplier method.

$$L(\omega, \lambda) = \omega^T \Sigma \omega - \lambda(\omega^T u - 1) \tag{23}$$

where λ is the Lagrange multiplier. We will then calculate the partial derivatives that we will specify equal to 0:

$$\begin{cases} \dfrac{\partial L}{\partial \omega} = 2\Sigma \omega - \lambda u = 0 \\[2mm] \dfrac{\partial L}{\partial \lambda} = \omega^T u - 1 = 0 \end{cases} \tag{24}$$

We will then draw ω into equation (25), after which we will replace it in equation(26).

$$2\Sigma \omega - \lambda u = 0 \implies \omega = \frac{1}{2}\lambda \Sigma^{-1} u \tag{25}$$

$$\omega^T u - 1 = 0 \implies \omega^T u = 1 \tag{26}$$

$$\frac{1}{2}\lambda u^T \Sigma^{-1} u = 1 \implies \lambda = \frac{2}{u^T \Sigma^- u} \tag{27}$$

(27) in (25) we have:

$$\omega = \frac{1}{2}\left(\frac{2}{u^T \Sigma^{-1} u}\right) \Sigma^{-1} u \quad \text{donc} \quad \omega = \frac{\Sigma^{-1} u}{u^T \Sigma^{-1} u} \tag{28}$$

The efficient portfolio is given by $P(\omega)$.

Our objective is to be able to determine the efficient frontier or at least express a function that allows us to determine the portfolio for a μ_0 target level of return. This problem can be formulated as follows:

$$\begin{cases} Min\ \sigma_P^2(\omega) = \omega^T \Sigma \omega \\ \omega^T \mu = \mu_0 \\ \omega^T u = 1 \end{cases} \tag{29}$$

The Lagrangian is given by the following expression:

$$L(\omega, \lambda_1, \lambda_2) = \omega^T \Sigma \omega - \lambda_1(\omega^T \mu - \mu_0) - \lambda_2(\omega^T u - 1) \tag{30}$$

The first-order conditions are:

$$\begin{cases} \dfrac{\partial L}{\partial \omega} = 2\Sigma\omega - \lambda_1\mu - \lambda_2 u = 0 \\[2mm] \dfrac{\partial L}{\partial \lambda_1} = \omega^T\mu - \mu_0 = 0 \\[2mm] \dfrac{\partial L}{\partial \lambda_2} = \omega^T u - 1 = 0 \end{cases} \tag{31}$$

$$2\Sigma\omega - \lambda_1\mu - \lambda_2 u = 0 \implies \omega = \frac{1}{2}\Sigma^{-1}(\lambda_1\mu + \lambda_2 u) \tag{32}$$

$$\omega^T\mu - \lambda_0 \implies \frac{1}{2}\lambda_1\mu^T\Sigma^{-1}\mu + \frac{1}{2}\lambda_2 u^T\Sigma^{-1}\mu = \mu_0 \tag{33}$$

$$\omega^T u - 1 = 0 \implies \frac{1}{2}\lambda_1\mu^T\Sigma^{-1}u + \frac{1}{2}\lambda_2 u^T\Sigma^{-1}u = 1 \tag{34}$$

To overcome this system, we define the following expressions: $A = u^T\Sigma^{-1}\mu$, $B = \mu^T\Sigma^{-1}\mu$ and $C = u^T\Sigma^{-1}u$ to make the rest of the operations more accommodating:

$$\begin{cases} A\lambda_1 + B\lambda_2 = 2\mu_0 \\ A\lambda_1 + C\lambda_2 = 2 \end{cases} \iff \begin{pmatrix} B & A \\ A & C \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 2\begin{pmatrix} \mu_0 \\ 1 \end{pmatrix} \tag{35}$$

This becomes an identity of the well-known linear algebra $Mx = b \implies x = M^{-1}b$ for

$$M = \begin{pmatrix} B & A \\ A & C \end{pmatrix} \tag{36}$$

$$M^{-1} = \begin{pmatrix} B & A \\ A & C \end{pmatrix}^{-1} = \frac{1}{BC - A^2}\begin{pmatrix} C & -A \\ -A & B \end{pmatrix} \tag{37}$$

According to the co-factors and

$$b = 2\begin{pmatrix} \mu_0 \\ 1 \end{pmatrix} \tag{38}$$

By designating $D = BC - A^2$ we finally get

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{2}{D}\begin{pmatrix} C & -A \\ -A & B \end{pmatrix}\begin{pmatrix} \mu_0 \\ 1 \end{pmatrix} = \frac{2}{D}\begin{pmatrix} C\mu_0 - A \\ -A\mu_0 + B \end{pmatrix} \tag{39}$$

which implies $\lambda_1 = 2\dfrac{-A + C\mu_0}{D}$ and $\lambda_2 = 2\dfrac{B - A\mu_0}{D}$. By replacing the expressions of λ in the expression of ω, you get:

$$\omega = \frac{1}{2}\Sigma^{-1}(\lambda_1\mu + \lambda_2 u) = \frac{1}{D}\Sigma^{-1}(-A\mu + B\mu) + \frac{1}{D}\Sigma^{-1}(C\mu - Au)\mu_0 \tag{40}$$

The efficient frontier function, which is really just an expression of σ_P^2 in function of μ_0 is:

$$\sigma_P^2(\mu_0) = \omega(\mu_0)^T\Sigma^{-1}\omega(\mu_0) \tag{41}$$

After determining the efficient frontier of a portfolio we will determine the tangent portfolio. This portfolio offers the best excess return per unit of risk, the best of sharpe ratio $\dfrac{\mu_P - R_P}{\sigma_P}$.

Determination of the tangent portfolio In order to find it, we reason by taking into account an additional asset (the risk-free asset), A_f. This asset provides a return R_f and a variance where risk is zero, and its covariance with other securities is zero, $Cov(R_f; R_j)=0$. The investor asks himself the same questions as above.

Suppose we call it ω_τ, the risky asset share and ω_f, the risk-free asset share of our portfolio, we should have:

$$\omega_\tau^T u + \omega_f = 1 \implies \omega_f = (1 - \omega_\tau^T u) \tag{42}$$

Also, since our risk-free assets have zero variance and zero correlation with other risky assets, the only risk in our new portfolio $P(\omega_\tau)$, comes from risky assets only. So we're going to solve the following problem:

$$\begin{cases} Min \ \sigma_P^2(\omega_\tau) = \omega^T \Sigma \omega_\tau \\ s/c \ \ \omega_\tau^T u + (1 - \omega_\tau^T u)R_f = \mu_0 \end{cases} \tag{43}$$

The Lagrangian of the program is as follows:

$$\mathcal{L}(\omega_\tau, \lambda) = \omega_\tau^T \Sigma \omega_\tau - \lambda(\omega_\tau^T(\mu - R_f u) + R_f + \mu_0) \tag{44}$$

The first-order conditions are:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial \omega_\tau} = 2\Sigma \omega_\tau - \lambda(\mu - R_f u) = 0 \\ \dfrac{\partial L}{\partial \lambda} = \omega_\tau^T(\mu - R_f u) + R_f - \mu_0 = 0 \end{cases} \tag{45}$$

We will as before determine an expression of ω_τ based on λ. From (45), we have:

$$\omega_\tau = -\frac{1}{2}\lambda \Sigma^{-1}(\mu - R_f u) \tag{46}$$

$$\omega_\tau^T(\mu - R_f u) = \mu_0 - R_f \tag{47}$$

which enables us to have:

$$-\frac{\lambda}{2}(\mu - R_f u)^T \Sigma^{-1}(\mu - R_f u) = (\mu_0 - R_f) \ \ \text{soit} \ \ \lambda = -\frac{2(\mu_0 - R_f)}{(\mu - R_f u)^T \Sigma^{-1}(\mu - R_f u)} \tag{48}$$

where the differences $(\mu - R_f)$ and $(\mu - R_f u)$ are respectively the excess returns first from the portfolio and then from the securities. The expression for ω_τ is:

$$\omega_\tau = (\mu_0 - R_f)\left(\frac{\Sigma^{-1}(\mu - R_f u)}{(\mu - R_f u)^T \Sigma^{-1}(\mu - R_f u)}\right) = \frac{\mu_0 - R_f}{(\mu - R_f u)^T} \tag{49}$$

This portfolio that we have just obtained $P(\omega_\tau)$ belongs, in fact, to a family of portfolios that we will see in the diagram below. The right-hand side of Capital Market Line (CML) whose most finite expression known thanks to the work of W.Sharpe et al is

$$\mu_p = R_f + \sigma_p\left[\frac{\mu_M - R_f}{\sigma_M}\right] \tag{50}$$

where μ_M and σ_M are the expectation and standard deviation of Market. The tangent portfolio will then graphically be the intersection of the Market line and the efficient frontier. It's the one that offers the best expected return, the best Sharpe ratio $\dfrac{\mu_p - R_f}{\sigma_p}$. By calculation, to obtain this portfolio, we will equalize the derivatives of the market line and the curve; then we will determine the ω that verifies this equation. But this method is tedious since the curve above shows us that the line and the curve intersect at one point, so that's where they must have the same securities. This observation allows us to conclude that at this point, $\omega_f=0$ and therefore the portfolio is entirely made up of risky assets i.e.

$\omega_\tau^T u = 1$ and by relying on our reasoning, we can write that at this point $\omega_\tau = \omega_s$ with ω_s the composition of the super-efficient portfolio. Formally at this point we have:

$$\omega_\tau = \omega_s = (\mu_0 - R_f)\frac{\Sigma^{-1}(\mu - R_f u)}{(\mu - R_f u)^T \Sigma^{-1}(\mu - R_f u)} \tag{51}$$

$$(\omega_s^T)u = \omega_\tau^T u = u^T \omega_s = (\mu_0 - R_f)\frac{u^T \Sigma^{-1}(\mu - R_f u)}{(\mu - R_f u)^T \Sigma^{-1}(\mu - R_f u)} = 1 \tag{52}$$

$$(\mu_0 - R_f) = \frac{(\mu - R_f u)^T \Sigma^{-1}(\mu - R_f u)}{u^T \Sigma^{-1}(\mu - R_f u)} \tag{53}$$

This new expression of excess portfolio return will be implemented in the expression of ω_s

$$\omega_s = (\mu_0 - R_f)\frac{\Sigma^{-1}(\mu - R_f u)}{(\mu - R_f u)^T \Sigma^{-1}(\mu - R_f u)} \tag{54}$$

$$= \left(\frac{(\mu - R_f u)^T \Sigma^{-1}(\mu - R_f u)}{u^T \Sigma^{-1}(\mu - R_f u)}\right)\frac{\Sigma^{-1}(\mu - R_f u)}{(\mu - R_f u)^T \Sigma^{-1}(\mu - R_f u)} \tag{55}$$

after simplification, we can conclude that the tangent portfolio $P(\omega_s)$, consists of:

$$\omega_s = \frac{\Sigma^{-1}(\mu - R_f u)}{u^T \Sigma^{-1}(\mu - R_f u)} \tag{56}$$

We finally see that the constitution of a Tangent portfolio, depends only on μ_0, therefore on the investor's preferences.

3   Presentation of the Downside Risk Method

As mentioned before, the Markowitz Mean-Variance model suffers from several flaws. The model is valid only if returns are normally distributed (which is not always the case), or if investors have stated preferences of the Mean-Variance type. Several researchers, such as Lee et al (2009), have pointed out that variance is a dubious measure of risk because it treats upper and lower returns the same way, or at a target value that may be the average return or a given return B called the "Benchmark" "Target Rate of Return". As a result, investors want to minimize below-average returns or generally returns below a target rate of return. As a result, the mean-variance model becomes unable to adequately capture the characteristics of portfolio returns and investors' perceptions of risk. Therefore, the use of downside risk measures seems to be necessary to better present investor preferences and to take into account the asymmetry of returns.

Downside risk is therefore an indicator that takes into account only undesirable (or negative) returns. It provides a synthetic measure of the frequency and intensity with which an investment has performed below a predetermined threshold return. It is defined as the probability of obtaining a return below a given value.

The objective of this model is to maximize the probability that the portfolio's return will exceed a certain minimum level of return, often referred to as the "benchmark threshold" or "disaster threshold" terminology: Roy's Safety-First measure, semi-variance, Lower Partial Moment, Value-at-Risk (VaR), Conditional Value at Risk (CVaR).

3.1   Choice of an Optimal Portfolio in the Context of Downside Risk: Case of Semi-Variance

In the Markowitz Mean-Variance method, the choice of an optimal portfolio is made by minimizing the variance or maximizing the mean. Downside risk is chosen by minimizing the semi-variance. Thus we will first present the mean-semi-variance model.

The mean-semivariance model The basis of the mean-semivariance approach is mainly due to Markowitz (1959) who tries to correct his error on the first method.

Let r_{it} be the returns observed over time $t=1,...,T$. The semi variance of return of asset i with respect to Benchmark B is given by the following expression:

$$\Sigma_{iB}^2 = \mathbb{E}\{[Min(r_i - B, 0)]^2\} = \frac{1}{T}\sum_{t=1}^{T}[Min(r_{it} - B, 0)]^2 \tag{57}$$

when $B = \mu$ then we speak of Markowitz semi-variance given by:

$$SV = \mathbb{E}\{[Min(r_i - \mu, 0)]^2\} = \frac{1}{T}\sum_{t=1}^{T}[Min(r_{it} - \mu, 0)]^2 \tag{58}$$

where μ is the average yield of the observations over time. The semi-covariance is given by:

$$\Sigma_{ijB} = \frac{1}{T}\sum_{t\in V}(r_{it} - B)(r_{jt} - B) \tag{59}$$

where V is the set of t indices such that $(R_{pt} - B) < 0$. More explicitly, the semi-covariance between assets i and j is given by:

$$\Sigma_{ij} = \mathbb{E}\{Min(r_i - B, 0) \times Min(r_j - B, 0)\} = \frac{1}{T}\sum_{t=1}^{T}[Min(r_{it} - B, 0) \times Min(r_{jt} - B, 0)] \tag{60}$$

This definition is generalizable to all Benchmarks and can be found in Estarda (2002,2007). Moreover, the Semi-covariance matrix whose coefficients are Σ_{ij} exogenous and symmetrical ($\Sigma_{ij} = \Sigma_{ji}$).

Choosing the optimal portfolio Here investors prefer assets with low semi-variance, i.e. assets with less negative skewness (i.e. assets below average).

These are $(\omega_1, \omega_2, ..., \omega_t, ..., \omega_m)$ the allocations assigned to the different assets that make up the portfolio. The return on this portfolio is a linear combinasion of the returns of the other assets. In other words, if r_{pt} is the return of portfolio P at time t, we have:

$$r_{pt} = \omega_1 r_{1t} + \omega_2 r_{2t} + ... + \omega_m r_{mt} \tag{61}$$

where r_{jt} is the return on asset j at time t.

Average-Semi-variance, the optimal portfolio is solution of the following minimization program:

$$\begin{cases} Min\ DSR\ (\omega_1, \omega_2, ..., \omega_m) = \frac{1}{T}\sum_{t=1}^{T}[min(r_{pt} - B, 0)]^2 \\ s/c\ \sum_{j=1}^{m}\omega_j\mu_j\ =\ E^*\ ;\ \sum_{j=1}^{m}\omega_j = 1 \end{cases} \tag{62}$$

where μ_j represents the estimated expected return on assets j, $j=1,...,m$ and E^* an expected return for the portfolio.

The optimization program (62) can have the following matrix posting:

$$\begin{cases} Min_\omega\ \omega^T M\omega \\ \omega^T\mu = E^*\ ;\ \omega^T\mathbf{1} = 1 \end{cases} \tag{63}$$

For the resolution of this program we will always use the Lagrange multiplier method and using the Athayde recursive minimization procedure for m assets with $R_{jt} = r_{jt} - B$.

A choice of ω_0 is used to start the procedure, an S_0 observation set is selected in which the weight of the ω_0 portfolio had negative deviations. The positive semi-variance matrix is then defined by:

$$M_0 = \frac{1}{T} \sum_{t \in S_0} \begin{bmatrix} R_{1t} \\ R_{2t} \\ \vdots \\ R_{mt} \end{bmatrix} \begin{bmatrix} R_{1t} R_{2t} \cdots R_{mt} \end{bmatrix} \tag{64}$$

$$= \frac{1}{T} \sum_{t \in S_0} \begin{bmatrix} (R_{1t})^2 & R_{1t}R_{2t} & \cdots & R_{1t}R_{mt} \\ R_{2t}R_{1t} & (R_{2t})^2 & \cdots & R_{2t}R_{mt} \\ \vdots & \vdots & \ddots & \vdots \\ R_{mt}R_{1t} & R_{mt}R_{2t} & \cdots & (R_{mt})^2 \end{bmatrix} \tag{65}$$

The next step is to find ω_1 which is the weight of the portfolio that solves the following minimization problem:

$$min_\omega \omega^T M_0 \omega \quad avec \quad \omega^T \mathbf{1} = 1 \tag{66}$$

where $\mathbf{1}$ is a vector and ω^T transposes it from the vector ω. Using the Lagrange Method, the solution to problem (66) is given by:

$$\omega_1 = \frac{M_0^{-1}}{\mathbf{1}^T M_0^{-1} \mathbf{1}} \tag{67}$$

With the new portfolio weight ω_1, a new positive semidefinite matrix M_1 is constructed as follows:

$$M_1 = \frac{1}{T} \sum_{t \in S_1} \begin{bmatrix} (R_{1t})^2 & R_{1t}R_{2t} & \cdots & R_{1t}R_{mt} \\ R_{2t}R_{1t} & (R_{2t})^2 & \cdots & R_{2t}R_{mt} \\ \vdots & \vdots & \ddots & \vdots \\ R_{mt}R_{1t} & R_{mt}R_{2t} & \cdots & (R_{mt})^2 \end{bmatrix} \tag{68}$$

The next step is to find ω_2 the portfolio weight that solves the following problem:

$$min_\omega \omega^T M_1 \omega \quad avec \quad \omega^T \mathbf{1} = 1 \tag{69}$$

As previously a solution to this problem is given by:

$$\omega_2 = \frac{M_1^{-1}}{\mathbf{1}^T M_1^{-1} \mathbf{1}} \tag{70}$$

The process is repeated to construct a M_t matrix until the first satisfactory M_F is obtained $M_F = M_{F+1}$. The weight of the optimal portfolio obtained by the DSR method is given by:

$$\omega_F = \frac{M_F^{-1}}{\mathbf{1}^T M_F^{-1} \mathbf{1}} \tag{71}$$

3.2   Efficient Frontier

Using the Lagrange process, the optimal convergent portfolio weight is given by:

$$\omega_{F+1} = \frac{\alpha E^* - \lambda}{\alpha\theta - \lambda^2} M_F^{-1} + \frac{\theta - \lambda E^*}{\alpha\theta - \lambda^2} M_F^{-1} \mathbf{1} \tag{72}$$

where $\alpha = \mathbf{1}^T M_F^{-1} \mathbf{1}$, $\lambda = \mu^T M_F^{-1} \mathbf{1}$ et $\theta = \mu^T M_F^{-1} \mu$

The return expectation is given by $\mu = \omega_{F+1}^T M_F$ and the efficient frontier is obtained as follows:

$$DSR(\omega_F) = \frac{\alpha(E^*)^2 - 2\lambda E^* + \theta}{\alpha\theta - \lambda^2} \tag{73}$$

4 Application to Securities of a Financial Market

4.1 Description of the Database

In order to empirically examine the differences between the mean-variance model and the portfolio optimization model in Downside Risk, we will use data from a global market index, the Morgan Stanley Capital Indices (MSCI) database (CAC40), which includes more than 71 stocks. Among these stocks, we used only 10 optimal stocks because they are characterized by asymmetric return distributions. The security symbols used are shown in the table below.

Table 1. Table of securities and their respective symbols

AC	Accor
AF	Air France
BN	DANONE
TA	TOTAL
OR	L'OREAL
RNO	RENAULT
ML	MICHELIN
UG	PEUGOT
SGE	SOCIÉTÉ GÉNÉRALE
SAF	SAGEM

4.2 Analysis of Collinearity

The analysis of the correlation between assets allows for portfolio diversification, i.e. combining negatively correlated assets to reduce the variance, without eroding the expected return.

Table 2. Pearson correlation matrix at the 5% threshold between assets

|	AC	AF	BN	TA	OR	RNO	ML	UG	SGE	SAF
AC	1.00	0.35	0.29	0.40	0.40	0.10	0.46	0.28	0.45	0.35
AF	0.40	1.00	0.52	0.38	0.35	0.39	0.35	0.27	0.23	0.42
BN	0.32	0.24	1.00	0.28	0.56	0.30	0.21	0.32	0.58	0.51
TA	0.30	0.45	0.46	1.00	0.25	0.50	0.67	0.24	0.45	0.17
OR	0.40	0.42	0.48	0.54	1.00	0.40	0.17	0.37	0.40	0.23
RNO	0.36	0.53	0.28	0.34	0.45	1.00	0.37	0.17	0.20	0.12
ML	0.24	0.29	0.17	0.47	0.22	0.37	1.00	0.25	0.37	0.41
UG	0.23	0.40	0.20	0.47	0.37	0.37	0.40	1.00	0.47	0.27
SGE	0.20	0.25	0.36	0.30	0.40	0.29	0.25	0.37	1.00	0.21
SAF	0.18	0.21	0.29	0.14	0.25	0.47	0.42	0.14	0.23	1.00

4.3 Results of the Mean-Variance Model

Table 3 below presents the different risk results according to the Mean-Variance model obtained by varying the expected return to the investor. It can be seen that risk is an increasing function of return.

The Mean-Variance (M-V) efficient frontier includes portfolios that represent the best relationship between return and risk. By varying the expected return, we obtain the Mean-Variance efficient frontier described by the graph below:

Table 3. Table presenting the risk results for the Mean-Variance model

	P1	P2	P3	P4	P5	P6	P7	P8	P9
Return	0,50%	0,60%	0,70%	0,80%	0,90%	1%	1,10%	1,20%	1,30%
Standard	3,352%	3,362%	3,368%	3,371%	3,72%	3,73%	3,782%	3,8%	3,86%
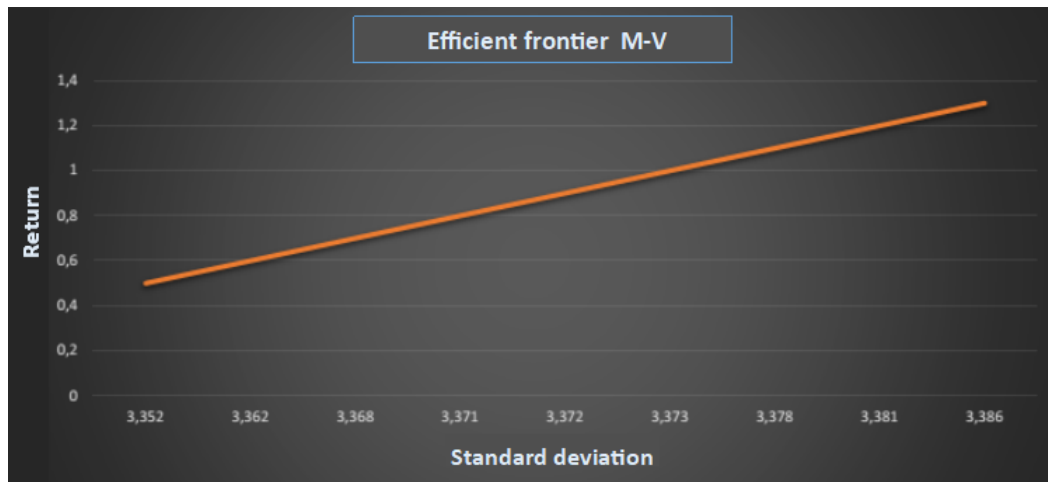
Figure 1. Smooth efficient frontier Mean-Variance

However, the mean-variance model is questionable because it considers variance as a measure of risk that is relevant only when the distribution of returns is normal. However, the results show that the returns on the selected securities are asymptotically distributed. Moreover, with variance, above-average returns are taken into account in the calculation of risk, which is contradictory to investors' perception of risk. In order to overcome this shortcoming, we propose the Downside Risk method whose optimization model is given above.

4.4   Results in the Context of Downside Risk

We begin our analysis by looking at the following Table 4. It shows the difference between the standard deviation and the Downside Risk Deviation (DSR). We use two target return rates, namely the risk-free rate and the zero value. The results obtained indicate that the Downside Risk for all target returns are lower than the standard deviations of all securities. The results also show that as the target rate of return increases, so does the risk. Although the Downside Risk deviation increases as the target rate of return increases, the results show that it always remains below average. There are not too many surprises because all these securities all have positive averages. SAGEM has a standard deviation of zero, so the Downside Risk is also zero.

Tables 5 and 6 below present the risk obtained by varying the expected returns. Note that by varying the expected returns desired by investors, the measure of Downside Risk for a given target rate of return is better relative to the standard deviation.

The level of risk measured by the Downside Risk deviation for $(\tau = 0)$ optimal portfolios is lower than the Mean-Variance and Mean-Semivariance models for $(\tau = R_F)$. The results imply that for the same level of expected return on optimal portfolios, the mean-semivariance model for $(\tau = 0)$ reduces risk by $(0, 22\%)$ relative to the mean-semivariance model for $(\tau = R_F)$ and (1.63%) relative to the mean-variance model.

With respect to the efficient frontier, Figure 2 shows that the change in the target rate of return has an impact on the efficient frontier. This graph shows the efficient frontiers obtained in the mean-semivariance

Table 4. Statistics showing standard deviation and Downside Risk

Secutities	average	standard $\times 10^{-3}$	$DSR \times 10^{-3}(\tau = R_F)$	$DRS \times 10^{-3}(\tau = 0)$
AC	0.05	10.00	8.50	6.23
AF	0.03	1.80	1.60	1.26
BN	0.01	5.70	4.60	3.52
TA	0.04	4.60	3.60	3.51
OR	0.03	17.70	15.60	13.57
RNO	0.19	2.73	1.73	1.32
ML	0.12	178.00	122.00	115.00
UG	0.15	1.10	1.01	0.85
SGE	0.20	2.94	1.18	1.04
SAF	0.25	0.00	0.00	0.00

Table 5. Table presenting the DSR for a target rate of return equal to 0

Return	0.50%	0.60%	0.70%	0.80%	0.90%	1%	1.10%	1.20%	1.30%
DSR ($\tau = 0$)	1.717	1.718	1.719	1.721	1.724	1.725	1.727	1.729	1.732

model relative to the two minimum acceptable rates of return [1]. We find that the efficient frontier moves to the right by increasing the benchmark rate of return from 0% to the risk-free rate. Indeed, the increase in the target rate of return leads to the choice of riskier portfolios for the same level of return. Such a result is consistent with the characteristics of Downside Risk.
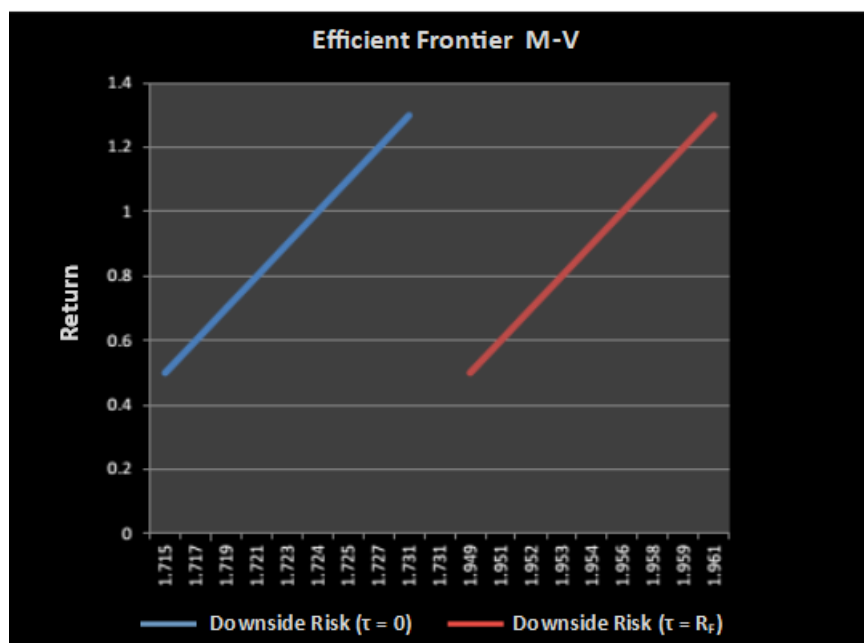

Figure 2. Efficient frontiers in the DSR framework

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[1] A minimum acceptable rate of return is the minimum profit an investor expects to make from an investment, taking into account the risks of the investment and the opportunity cost of undertaking it instead of other investments. An investment has been a successful one if the actual rate of return is above the minimum acceptable rate of return. If it is below, it's seen as an unsuccessful investment and you might, as an investor, pull out of the investment.

Table 6. Table showing the DSR for the target rate of return equal to R_f

Return	0.50%	0.60%	0.70%	0.80%	0.90%	1%	1.10%	1.20%	1.30%
DSR ($\tau = R_F$)	1.941	1.951	1.954	1.955	1.956	1.957	1.958	1.962	1.975

  Figure 3 shows the difference between the mean-variance efficient frontier and the mean-semivariance (M-SV) efficient frontiers for the target rates of return considered. This graph shows that the mean-semivariance efficient frontier clearly dominates the mean-variance frontier derived from the mean-variance model. Indeed, for a given expected rate of return, mean-semivariance efficient portfolios are less risky than those derived by the mean-variance model. Furthermore, we find that the observed gap between the mean-semivariance efficient frontier and the mean-variance efficient frontier decreases with the target rate of return. Indeed, the difference observed between these two types of frontiers is attributed to the asymmetry in securities returns and investors' perceptions of risk not captured by the mean-variance model.
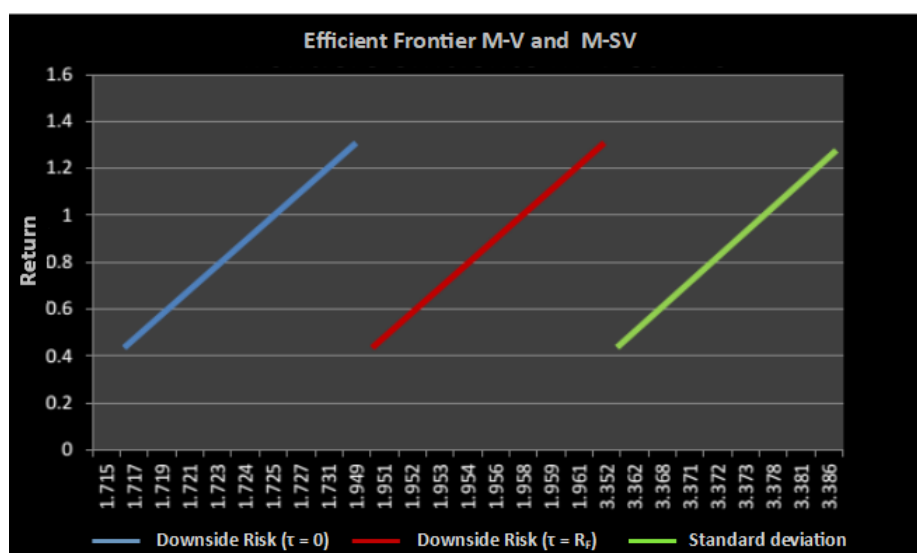
Figure 3. Efficient frontiers of the two models

5   Conclusion

In this article we have realized and found an efficient portfolio and a smooth efficient frontier by the downside risk method. After determining an efficient portfolio by the mean-variance model, we found that the mean-variance model is not a good risk measurement for a portfolio. The Markowitz Mean-Variance model uses variance as a risk measurement, and since variance is symmetric (i.e., the true value treats positive data as well as negative data), we introduced the Downside Risk measure to address this issue. This model gives hope to investors because it takes into account only returns below a target rate of return to predict risk.

  After application, it can be seen that the efficient Downside Risk portfolio is less risky than the one obtained by minimizing the variance. This is in line with our expectations.

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