Polynomial Inequalities in Regions Bounded by Piecewise Asymptotically Conformal Curve with Nonzero Angles in the Bergman Space

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Abstract We continue the study of estimates of algebraic polynomials in regions bounded by a piecewise asymptotically conformal curve with interior non-zero angles in the weighted Bergman space.

Keywords: Algebraic polynomials, Conformal mapping, Asymptotically conformal curve.

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1 Introduction and Main Results

Let $G \subset \mathbb{C}$ be a finite region, with $0 \in G$, bounded by a Jordan curve $L := \partial G$, $\Omega := \text{ext} L := \mathbb{C} \setminus G$, where $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, $\Delta := \{w: |w| > 1\}$ and let $\wp_n$ denote the class of arbitrary algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N}$. Let $w = \Phi(z)$ be the univalent conformal mapping of $\Omega$ onto the $\Delta$ normalized by $\Phi(\infty) = \infty$, $\lim_{z \to \infty} \frac{\Phi(z)}{z} > 0$, and $\Psi := \Phi^{-1}$. For $t \geq 1$, $z \in \mathbb{C}$, we set:

$L_t := \{z: |\Phi(z)| = t\}$ ($L_1 \equiv L$), $G_t := \text{int} L_t$, $\Omega_t := \text{ext} L_t$.

Let $(z_j)_{j=1}^m$ be a fixed system of distinct points on curve $L$, located in the positive direction. For some fixed $R_0$, $1 < R_0 < \infty$, and $z \in G_{R_0}$, consider a so-called generalized Jacobi weight function $h(z)$ being defined as follows:

$h(z) := h_0(z) \prod_{j=1}^m |z - z_j|^\gamma_j$, $z \in G_{R_0}$, (1.1)

where $\gamma_j > -2$, for all $j = 1, 2, ..., m$, and the function $h_0$ is uniformly separated from zero in $G_{R_0}$, i.e. there exists a constant $c_0 := c_0(G_{R_0}) > 0$ such that, for all $z \in G_{R_0}$

$h_0(z) \geq c_0 > 0$.

For any $p > 0$ and for Jordan region $G$, let’s define:

$\|P_n\|_p := \|P_n\|_{A_p(h,G)} := \left( \int \int_G h(z) |P_n(z)|^p \, d\sigma_z \right)^{1/p} < \infty, \ 0 < p < \infty; \ (1.2)$

$\|P_n\|_\infty := \|P_n\|_{A_{\infty}(h,G)} := \|P_n\|_{C(G)}, \ p = \infty,$

where $\sigma_z$ is the two-dimensional Lebesgue measure.

In this work, we continue the study of the following Nikolskii-type inequality:

$\|P_n\|_\infty \leq c_1 \lambda_n(G, h, p) \|P_n\|_p$, (1.3)
where $c_1 = c_1(G, h, p) > 0$ is a constant independent of $n$ and $P_n$, and $\lambda_0(G, h, p) \to \infty$, $n \to \infty$, depending on the geometrical properties of region $G$, weight function $h$ and of $p$. The estimate of (1.3)-type for some $(G, p, h)$ was investigated in [27, pp.122-133], [17], [26, Sect.5.3], [32], [15], [2]-[8] (see, also, references therein) and others. Further, analogous of (1.3) for some regions and the weight function $h(z)$ were obtained: in [8] for $p > 1$ and for regions bounded by piecewise Dini-smooth boundary without cusps; in [11] for $p > 0$ and for regions bounded by quasiconformal curve; in [7] for $p > 1$ and for regions bounded by piecewise smooth curve without cusps; in [10] for $p > 0$ and for regions bounded by asymptotically conformal curve; in [16] for $p > 0$ and for regions bounded by piecewise smooth curves with interior (zero or nonzero) angles, in [12] for $p > 0$ and for regions bounded by piecewise asymptotically conformal curve having cusps and others.

In this work, we investigate similar problems for $z \in \mathcal{G}$ in regions bounded by piecewise asymptotically conformal curves having interior nonzero zero angles and for weight function $h(z)$, defined in (1.1) and for $p > 0$.

Now, we begin to give some definitions and notations.

Following [24, p.97], [28], the Jordan curve (or arc) $L$ is called $K$-quasiconformal ($K \geq 1$), if there is a $K$-quasiconformal mapping $f$ of the region $D \supset L$ such that $f(L)$ is a circle (or line segment).

Let $S$ be a Jordan curve and $z = z(s)$, $s \in [0, |S|]$, $|S| := mes S$, denote the natural representation of $S$. Let $z_1, z_2 \in S$ be an arbitrary points and $S(z_1, z_2) \subset S$ denotes the subarc of $S$ of shorter diameter with endpoints $z_1$ and $z_2$. The curve $S$ is a quasicircle if and only if the quantity

$$\sup_{z_1, z_2 \in S} \frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|}$$

is bounded. Following to Lesley [25], the curve $S$ to be said "$c$-quasiconformal", if the quantity (1.4) bounded by positive constant $c$, independent from points $z_1, z_2$ and $z$. At the literature it is possible to find various functional definitions of the quasiconformal curves (see, for example, [29, pp.286-294], [24, p.105], [13, p.81], [30, p.107]).

The Jordan curve $S$ is called asymptotically conformal [19], [30], if

$$\sup_{z_1, z_2 \in S} \frac{|z_1 - z| + |z - z_2|}{|z_1 - z_2|} \to 1, \quad |z_1 - z_2| \to 0.$$  

We will denote this class as AC, and will write $G \in AC$, if $L := \partial G \in AC$.

The asymptotically conformal curves occupy a special place in the problems of the geometric theory of functions of a complex variable. These curves in various problems have been studied by Anderson, Becker and Lesley [14], Dyn’kin [20], Pommerenke, Warschawski [31], Gutlyanskii, Ryazanov [21], [22], [23] and others. According to the geometric criteria of quasiconformality of the curves ([13, p.81], [30, p.107]), every asymptotically conformal curve is a quasicircle. Every smooth curve is asymptotically conformal but corners are not allowed. It is well known that quasicircles can be non-rectifiable (see, for example, [18], [24, p.104]). The same is true for asymptotically conformal curves.

A Jordan arc $\ell$ is called asymptotically conformal arc, when $\ell$ is a part of some asymptotically conformal curve.

Now, we define a new class of regions bounded by piecewise asymptotically conformal curve having exterior nonzero "angles" at the connecting points of boundary arcs.

Throughout this work, we will assume that $p > 0$ and the constants $c, c_0, c_1, c_2, \ldots$ are positive and constants $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ are sufficiently small positive (generally, are different in different relations), which depends on $G$ in general and, on parameters inessential for the argument, otherwise, the dependence will be explicitly stated. Also note that, for any $k \geq 0$ and $m > k$, notation $j = \overline{k, m}$ denotes $j = k, k + 1, \ldots, m$.

Now, let’s introduce ’special angles’ on $L$.

**Definition 1.1.** We say that a Jordan region $G \in PAC(\nu_1, \ldots, \nu_m)$, $0 < \nu_j < 2$, $j = \overline{1, m}$, if $L = \partial G$ consists of the union of finite asymptotically conformal arcs $\{L_j\}_{j=1}^m$, connected at the points $\{z_j\}_{j=0}^m \in L$ such that in $z_0$ - $L$ locally asymptotically conformal and for any $z_j \in L, j = \overline{1, m}$, where two arcs $L_{j-1}$ and $L_j$ meet, there exist $r_j := r_j(L, z_j) > 0$ and $\nu_j := \nu_j(L, z_j)$, $0 < \nu_j < 2$, such that for some $0 \leq \theta_0 < 2\pi$ a closed maximal circular sector $S(z_j ; r_j, \nu_j) := \{\zeta : \zeta = z_j + r_j e^{i\theta}, \theta_0 < \theta < \theta_0 + \nu_j\}$ of radius $r_j$ and opening $\nu_j\pi$ lies in $G = \text{int}L$ with vertex at $z_j$. 

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Clearly, that \( PAC(\nu_1) \subseteq PAC(\nu_2) \), if \( \nu_2 \geq \nu_1 \).

**Definition 1.2.** We say that a Jordan region \( G \in PAC(\nu) \), if \( G \in PAC(\nu_1, ..., \nu_m) \), \( 0 < \nu_j < 2 \), \( j = 1, m \), where \( \nu = \min(\nu_j : 0 < \nu_j < 2, \ j = 1, m) \).

It is clear from Definition 1.1 (1.2), that each region \( G \in PAC(\nu_1, ..., \nu_m) \), \( 0 < \nu_1, ..., \nu_m < 2 \), \( (G \in PAC(\nu)) \) may have 'singularity' at the boundary points \( \{z_i\}_{i=1}^{m} \in L \). If it does not have such 'singularity' (in this case we put \( \nu_i = 1, i = 1, m \), then it is written as \( G \in AC \).

Throughout this work, we will assume that the points \( \{z_i\}_{i=1}^{m} \in L \) defined in (1.1) and \( \{\xi_i\}_{i=1}^{m} \in L \) defined in Definition 1.1 (1.2) coincide. Without the loss of generality, we also will assume that the points \( \{z_i\}_{i=1}^{m} \in L \) are ordered in the positive direction on the curve \( L \).

We state our new results. Assume that the curve \( L \) have 'singularity' on the boundary points \( \{z_i\}_{i=1}^{m} \), i.e., \( \nu_i < 1 \), for all \( i = 1, m \), and the weight function \( h \) have 'singularity' at the same points, i.e., \( \gamma_i \neq 0 \) for some \( i = 1, m \). In this case, we have the following:

**Theorem 1.1.** Let \( p > 0 \). Suppose that \( G \in PAC(\nu_1, ..., \nu_m) \) for some \( 0 < \nu_1, ..., \nu_m < 1 \); \( h(z) \) defined as in (1.1). Then, for any \( P_n \in \wp_n \), \( n \in N \), and arbitrarily small \( \varepsilon > 0 \), there exists \( c_1 = c_1(G, p, \gamma_j) > 0 \) such that

\[
\|P_n\|_\infty \leq c_1(n + 1)^{2\tilde{\nu}} \|P_n\|_p + \varepsilon \|P_n\|_p, \tag{1.6}
\]

where \( \tilde{\nu} := \max\{0, \gamma_i\} \) and \( \tilde{\nu} := \min\{\nu_i\}, \ i = 1, m \).

**Theorem 1.2.** Let \( p > 0 \). Suppose that \( G \in PAC(\nu_1, ..., \nu_m) \) for some \( 0 < \nu_1, ..., \nu_m < 1 \); \( h(z) \) defined as in (1.1). Then, for any \( P_n \in \wp_n \), \( n \in N \), and arbitrarily small \( \varepsilon > 0 \), there exists \( c_2 = c_2(G, p, \gamma_j) > 0 \) such that

\[
|P_n(z_j)| \leq c_2\mu_n \|P_n\|_p, \tag{1.7}
\]

where

\[
\mu_n := \begin{cases} 
(n^2 + \nu_1^2)^{1/2}, & \text{if } \gamma_j > \frac{1}{2-\nu_j} - 2 - \varepsilon, \\
(n \ln n)^{1/2}, & \text{if } \gamma_j = \frac{1}{2-\nu_j} - 2 - \varepsilon, \\
n^{1/2}, & \text{if } 2 - \gamma_j < \frac{1}{2-\nu_j} - 2 - \varepsilon.
\end{cases}
\]

The sharpness of the estimations (1.6) and (1.7) can be discussed by comparing them with the following result:

**Remark 1.1.** ([9, Theorem 1.1.5], [2]) For any \( n \in N \) there exists a polynomials \( Q_n^*, T_n^* \in \wp_n \) such that for unit disk \( B \) and weight function \( h^*(z) = |z - z_1|^2 \) the following is true:

\[
|Q_n^*(z)| \geq c_0n \|Q_n^*\|_{A_2(B)}, \quad \text{for all } z \in B; \\
|T_n^*(z)| \geq c_1n^2 \|T_n^*\|_{A_2(h^*; B)};
\]

2 **Some Auxiliary Results**

Throughout this work, for the nonnegative functions \( a > 0 \) and \( b > 0 \), we shall use the notations “\( a \preceq b \)” (order inequality), if \( a \leq cb \) and “\( a \succeq b \)” are equivalent to \( c_1a \leq b \leq c_2a \) for some constants \( c, c_1, c_2 \) (independent of \( a \) and \( b \)), respectively.

**Lemma 2.1.** ([1]) Let \( L \) be a \( K \)-quasiconformal curve, \( z_1, z_2, z_3 \in \Omega \cap \{z : |z - z_1| \leq d(z_1, L_\alpha)\}; \ w_j = \Phi(z_j), \ j = 1, 2, 3 \). Then

a) The statements \( |z_1 - z_2| \preceq |z_1 - z_3| \) and \( |w_1 - w_2| \preceq |w_1 - w_3| \) are equivalent.

So are \( |z_1 - z_2| \succeq |z_1 - z_3| \) and \( |w_1 - w_2| \succeq |w_1 - w_3| \);

b) If \( |z_1 - z_2| \preceq |z_1 - z_3| \), then

\[
\frac{|w_1 - w_3|}{|w_1 - w_2|} \preceq \frac{|z_1 - z_3|}{|z_1 - z_2|} \preceq \frac{|w_1 - w_3|}{|w_1 - w_2|}^{K^2},
\]

where \( 0 < r_0 < 1 \), \( R_0 := r_0^{-1} \) are constants, depending on \( G \).
According to (3.1) - (3.5), we have:

\[ \|P_n\|_{A_p(h,G_R^n)} \leq \tilde{R}^{n+\frac{1}{2}} \|P_n\|_{A_p(h,G)} , \quad p > 0, \]

(2.1)

where \( \tilde{R} = 1 + c(R - 1) \) and \( c \) is independent from \( n \) and \( R \).

### 3 Proof of Theorems

#### 3.1 Proof of Theorem 1.1

**Proof.** Suppose that \( G \in \text{PAC}(\nu_1,\nu_2) \) for some \( 0 < \nu_1,\nu_2 < 1 \) and \( h(z) \) is defined as in (1.1). Let \( \{\xi_j\}, \, 1 \leq j \leq m \leq n, \) be the zeros (if any exist) of \( P_n(z) \) lying on \( \Omega \). Let’s define the function Blaschke with respect to the zeros \( \{\xi_j\} \) of the polynomial \( P_n(z) \):

\[ \tilde{B}_j(z) := \frac{\Phi(z) - \Phi(\xi_j)}{1 - \Phi(\xi_j)\Phi(z)} , \quad z \in \Omega, \]

(3.1)

and let

\[ B_m(z) := \prod_{j=1}^{m} \tilde{B}_j(z) , \quad z \in \Omega. \]

(3.2)

It is easy that the

\[ B_m(\xi_j) = 0, \quad |B_m(z)| \equiv 1, \quad z \in L; \quad |B_m(z)| < 1, \quad z \in \Omega. \]

(3.3)

Then, for each \( \varepsilon_1, \, 0 < \varepsilon_1 < 1, \) there exists a circle \( \{ w : |w| = R_1 := 1 + \varepsilon_2, \, 0 < \varepsilon_2 < \frac{\varepsilon_1}{n} \} \) such that for any \( j = 1,2, \) the following holds:

\[ |\tilde{B}_j(\zeta)| > 1 - \varepsilon_2, \quad \zeta \in L_{R_1}. \]

(3.4)

So, from (3.2), we get:

\[ |B_m(\zeta)| > (1 - \varepsilon_2)^m \geq 1, \quad \zeta \in L_{R_1}. \]

For any \( p > 0 \) and \( z \in \Omega \) let us set:

\[ Q_{n,p}(z) := \left[ \frac{P_n(z)}{B_m(z)^{\Phi^{n+1}(z)}} \right]^{p/2}. \]

(3.5)

The function \( Q_{n,p}(z) \) is analytic in \( \Omega \), continuous on \( \overline{\Omega} \), \( Q_{n,p}(\infty) = 0 \) and does not have zeros in \( \Omega \). We take an arbitrary continuous branch of the \( Q_{n,p}(z) \) and for this branch, we maintain the same designation. According to Cauchy integral representation for the unbounded region \( \Omega \), we have:

\[ Q_{n,p}(z) = -\frac{1}{2\pi i} \int_{L_{R_1}} Q_{n,p}(\zeta) \frac{d\zeta}{\zeta - z} , \quad z \in \Omega_{R_1}. \]

(3.6)

According to (3.1) - (3.5), we have:

\[ |P_n(z)|^{p/2} = \frac{|B_m(z)^{\Phi^{n+1}(z)}(z)|^\frac{p}{2}}{2\pi d(z,L_{R_1})} \int_{L_{R_1}} \left| \frac{P_n(\zeta)}{B_m(\zeta)^{\Phi^{n+1}(\zeta)}} \right|^{p/2} |d\zeta| \]

(3.7)

\[ \leq |\Phi^{n+1}(z)|^\frac{p}{2} \int_{L_{R_1}} |P_n(\zeta)|^{p/2} |\zeta - z|. \]
Multiplying the numerator and the denominator of the last integrand by $h^{1/2}(\zeta)$, applying the Hölder inequality, we obtain:

$$
\left( \int_{R_n} |P_n(\zeta)|^2 \, d\zeta \right)^2 \leq \int_{|t|=R_1} h(\Psi(t)) \left| P_n(\Psi(t)) \right|^p \left| \Psi'(t) \right|^2 \, dt \cdot \int_{|t|=R_1} \frac{|dt|}{h(\Psi(t)) |\Psi(t) - \Psi(w)|^2} \tag{3.8}
$$

$$
\leq \int_{|t|=R_1} h(\Psi(t)) \left| P_n(\Psi(t)) \right|^p \left| \Psi'(t) \right|^2 \, dt \cdot \int_{|t|=R_1} \frac{|dt|}{h(\Psi(t)) |\Psi(t) - \Psi(w)|^2}
$$

$$
= \int_{|t|=R_1} |f_{n,p}(t)|^p \, dt \cdot \int_{|t|=R_1} \frac{|dt|}{h(\Psi(t)) |\Psi(t) - \Psi(w)|^2} =: A_n \cdot D_n(w),
$$

where $f_{n,p}(t) := h^{1/2}(\Psi(t)) P_n(\Psi(t)) \left( \Psi'(t) \right)^{1/2}$, $|t| = R_1$.

For the estimate integral $A_n$, we divide the circle $|t| = R_1$ into $n$ equal parts $\delta_n$ with $\text{mes} \delta_n = \frac{2\pi R_1}{n}$ and by applying the mean value theorem, we get:

$$
A_n := \int_{|t|=R_1} |f_{n,p}(t)|^p \, dt
$$

$$
= \sum_{k=1}^{n} \int_{\delta_k} |f_{n,p}(t)|^p \, dt = \sum_{k=1}^{n} \left| f_{n,p}(t') \right|^p \text{mes} \delta_k, \quad t' \in \delta_k.
$$

On the other hand, by applying mean value estimation

$$
\left| f_{n,p}(t') \right|^p \leq \frac{1}{\pi \left( |t'|-1 \right)^2} \int_{|t'-t'|=1} |f_{n,p}(\xi)|^p \, d\sigma, \quad \xi \in \delta_k.
$$

we obtain:

$$
(A_n)^2 \leq \sum_{k=1}^{n} \frac{\text{mes} \delta_k}{\pi \left( |t'|-1 \right)^2} \int_{|t'-t'|=1} |f_{n,p}(\xi)|^p \, d\sigma, \quad t' \in \delta_k.
$$

By taking into account, at most two of the discs with center $t'$ are intersecting, we have:

$$
A_n \lesssim \frac{\text{mes} \delta_1}{(|\xi|-1)^2} \int_{|\xi|<R} |f_{n,p}(\xi)|^p \, d\sigma \lesssim n \cdot \int_{1<|\xi|<R} |f_{n,p}(\xi)|^p \, d\sigma.
$$

According to Lemma 2.3, for $A_n$ we get:

$$
A_n \lesssim n \int_{G_n \setminus G} h(\zeta) |P_n(\zeta)|^p \, d\sigma \lesssim n \cdot \|P_n\|_{L^p}^p.
$$

To estimate the integral $D_n(w)$, denoted by $w_j := \Phi(z_j)$, $\varphi_j := \arg w_j$, for any fixed $\rho > 1$, we introduce:

$$
\Delta_1(\rho) := \left\{ t = re^{i\theta} : r > \rho, \quad \frac{\varphi_0 + \varphi_1}{2} \leq \theta < \frac{\varphi_1 + \varphi_2}{2} \right\},
$$

$$
\Delta_2(\rho) := \left\{ t = re^{i\theta} : r > \rho, \quad \frac{\varphi_1 + \varphi_2}{2} \leq \theta < \frac{\varphi_1 + \varphi_0}{2} \right\};
$$

$$
\Delta_j := \Delta_j(1), \quad \Omega_j := \Psi(\Delta_j), \quad \Omega_j^\rho := \Psi(\Delta_j(\rho));
$$

$$
L_j := L \cap \Omega_j^\rho, \quad L_\rho := L_\rho \cap \Omega_j^\rho, \quad j = 1, 2; \quad L = L^1 \cup L^2, \quad L_\rho = L_\rho^1 \cup L_\rho^2.
$$

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Under these notations, from (3.8) for the $D_n(w)$, we get:

$$D_n(w) = \int_{|t|=R_1} \frac{|dt|}{h(\Phi(t)) |\Psi(t) - \Psi(w)|^2}$$

\[\sum_{j=1}^{2} \int_{\Phi(L^j_{R_1})} \frac{|dt|}{\prod_{j=1}^{2} |\Phi(t) - \Phi(w_j)| \prod_{j=1}^{2} |\Psi(t) - \Psi(w)|^2} = \sum_{j=1}^{2} D_{n,j}(w),\]

since the points $\{z_j\}_{j=1}^{2} \in L$ are distinct. So, we need to evaluate the $D_{n,j}(w)$. For this, we take $z \in L_R$ and introduce the notations:

$$\Phi(L_{R_1}) = \bigcup_{j=1}^{2} \Phi(L^j_{R_1}) = \bigcup_{j=1}^{2} \Phi(L^j_{R_1}) = \bigcup_{j=1}^{2} K^j_i(R_1),$$

where

$$K^j_i(R_1) := \{ t \in \Phi(L^j_{R_1}) : |t - w_j| < c_i \}$$

Analogously,

$$\Phi(L_R) = \bigcup_{j=1}^{2} \Phi(L^j_R) = \bigcup_{j=1}^{2} \Phi(L^j_R) = \bigcup_{j=1}^{2} K^j_i(R),$$

where

$$K^j_i(R) := \{ t \in \Phi(L^j_R) : |t - w_j| < 2c_i \}$$

$$K^j_i(R) := \Phi(L^j_R) \setminus K^j_i(R), j = 1, 2.$$

Then, after these definitions, taking arbitrary fixed $w = \Phi(z) \in \Phi(L_R)$, the quantity $D_{n,j}(w)$ can be written as follows:

$$D_{n,j}(w) = \sum_{i=1}^{2} \int_{K^j_i(R_1)} \frac{|dt|}{|\Phi(t) - \Phi(w_j)| |\Psi(t) - \Psi(w)|^2} = \sum_{i=1}^{2} D^j_{n,j}(w)$$

The quantity $D^j_{n,j}(w)$ we shall estimate for each $i = 1, 2$ and $j = 1, 2$ in cases separately, depending of location of the $w \in \Phi(L_R)$. Let $\varepsilon > 0$ arbitrary small fixed number.

Case 1. Let $w \in \Phi(L^j_{R_1})$.

According to the above notations, we will make evaluations for case $w \in K^j_i(R)$ for each $i = 1, 2, 3$. 1.1) Let $w \in K^j_i(R)$. In this case, we will estimate the quantity

$$D_{n,1}(w) = \sum_{i=1}^{2} \int_{K^j_i(R_1)} \frac{|dt|}{|\Phi(t) - \Phi(w_j)| |\Psi(t) - \Psi(w)|^2} = \sum_{i=1}^{2} D^j_{n,1}(w)$$

for $\gamma_1 \geq 0$ and $\gamma_1 < 0$ separately.

For each $i = 1, 2$ and $j = 1, 2$ we put: $K^j_{i,1}(R_1) := \{ t \in \Phi(L^j_{R_1}) : |t - w_j| \geq |t - w| \}$, $K^j_{i,2}(R_1) := K^j_i(R_1) \setminus K^j_{i,1}(R_1)$. 

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1.1.1) If $\gamma_1 \geq 0$, then

$$
D_{n,1}^1(w) = \int_{K_1^i(R_1)} \frac{|dt|}{|\psi(t) - \psi(w_1)|^{\gamma_1} |\psi(t) - \psi(w)|^2}
$$

$$
= \int_{K_1^i(R_1)} \frac{|dt|}{|\psi(t) - \psi(w)|^{2+\gamma_1}} + \int_{K_{i,2}^i(R_1)} \frac{|dt|}{|\psi(t) - \psi(w_1)|^{2+\gamma_1}}
$$

$$
= : D_{n,1}^{1,1}(w) + D_{n,1}^{1,2}(w).
$$

Since $G \in PAC(\nu_1, \nu_2)$ for some $0 < \nu_1, \nu_2 < 1$, according to [25], $\psi \in \text{Lip}(\nu_i)$ and $\Phi \in \text{Lip}_{\frac{1}{2-\nu_i}}$, $i = 1, 2$, in a some fixed neighborhood of point $z_j$. Therefore, we get:

$$
D_{n,1}^{1,1}(w) \leq \int_{K_{i,1}^i(R_1)} \frac{|dt|}{|t - w|^{(2+\gamma_1)(2-\nu_1)}} \leq n^{(2+\gamma_1)(2-\nu_1)-1},
$$

and

$$
D_{n,1}^{1,2}(w) \leq \int_{K_{i,2}^i(R_1)} \frac{|dt|}{|t - w_1|^{(2+\gamma_1)(2-\nu_1)}} \leq n^{(2+\gamma_1)(2-\nu_1)-1},
$$

If $\gamma_1 < 0$, then

$$
D_{n,1}^1(w) = \int_{K_i^1(R_1)} \frac{|\psi(t) - \psi(w_1)|^{(-\gamma_1)} |dt|}{|\psi(t) - \psi(w)|^2}
$$

$$
\leq \int_{K_i^1(R_1)} \frac{|dt|}{|t - w|^{2-\nu_1}} \leq \int_{K_i^1(R_1)} \frac{|dt|}{|t - w|^{2-\nu_1}} \leq n^{2(2-\nu_1)-1}.
$$

1.1.2) If $\gamma_1 \geq 0$, then

$$
D_{n,1}^2(w) = \int_{K_i^1(R_1)} \frac{|dt|}{|\psi(t) - \psi(w_1)|^{\gamma_1} |\psi(t) - \psi(w)|^2}
$$

$$
= \int_{K_{i,1}^j(R_1)} \frac{|dt|}{|\psi(t) - \psi(w)|^{2+\gamma_1}} + \int_{K_{i,2}^j(R_1)} \frac{|dt|}{|\psi(t) - \psi(w_1)|^{2+\gamma_1}}
$$

$$
= : D_{n,1}^{2,1}(w) + D_{n,1}^{2,2}(w).
$$

and, so from Lemma 2.1 and 2.2, we get:

$$
D_{n,1}^{2,1}(w) \leq \int_{K_{i,1}^j(R_1)} \frac{|dt|}{|t - w|^{(2+\gamma_1)(2-\nu_1)}} \leq n^{(2+\gamma_1)(2-\nu_1)-1},
$$

and

$$
D_{n,1}^{2,2}(w) \leq 1.
$$

Therefore, from (3.19)-(3.21) for $\gamma_1 \geq 0$, we have:

$$
D_{n,1}^2(w) \leq n^{(2+\gamma_1)(2-\nu_1)-1}.
$$

For $\gamma_1 < 0$ from (3.14), we have:

$$
D_{n,1}^2(w) = \int_{K_i^1(R_1)} \frac{|\psi(t) - \psi(w_1)|^{(-\gamma_1)} |dt|}{|\psi(t) - \psi(w)|^2}
$$

(3.23)
\[
\lesssim \int_{K^1_{1,2}(R)} \frac{|dt|}{|t-w|^2(1+\varepsilon)} \leq n^{1+\varepsilon}, \quad \forall \varepsilon > 0.
\]

1.2) Let \( w \in K^2_2(R) \).

1.2.1) For any \( \gamma_1 > -2 \)

\[
D^1_{n,1}(w) = \int_{K^1_{1,2}(R)} \frac{|dt|}{|\psi(t) - \psi(w)|^{2+\gamma_1}} + \int_{K^2_{1,2}(R)} \frac{|dt|}{|\psi(t) - \psi(w_1)|^{2+\gamma_1}}
\]

and so, according to Lemmas 2.1 and 2.2, we obtain:

\[
D^1_{n,1}(w) \lesssim \int_{K^1_{1,2}(R)} \frac{|dt|}{|\psi(t) - \psi(w)|^{2+\gamma_1}} \leq 1,
\]

and

\[
D^1_{n,2}(w) \lesssim \int_{K^2_{1,2}(R)} \frac{|dt|}{|t-w|^{2+\gamma_1}(2-\nu_1)} \leq n^{(2+\gamma_1)(2-\nu_1)-1}.
\]

1.2.2) For any \( \gamma_1 > -2 \), according to Lemmas 2.1 and 2.2, we have:

\[
D^2_{n,1}(w) \lesssim \int_{K^1_{1,2}(R)} \frac{|dt|}{|\psi(t) - \psi(w)|^{2+\gamma_1}} + \int_{K^2_{1,2}(R)} \frac{|dt|}{|\psi(t) - \psi(w_1)|^{2+\gamma_1}}
\]

\[
\leq \int_{K^2_{1,2}(R)} \frac{|dt|}{|t-w|^{2+\gamma_1}+1} + 1 \leq n^{(2+\gamma_1)(1+\varepsilon)-1}, \quad \forall \varepsilon > 0.
\]

Combining estimates (3.14)-(3.26), for \( w \in \Phi(L_R) \), we have:

\[
D_{n,1} \lesssim n^{(2+\gamma_1)(2-\nu_1)-1+\varepsilon}, \quad \gamma_1 := \max \{0; \gamma_1\}.
\]

Case 2. Let \( w \in \Phi(L_R) \). Analogously to the Case 1, we will obtain estimates for \( w \in K^2_{2}(R) \) and \( w \in K^2_{2}(R) \)

\[
D_{n,2}(w) \lesssim n^{(2+\gamma_2)(2-\nu_2)-1+\varepsilon}, \quad \gamma_2 := \max \{0; \gamma_2\}
\]

Therefore, comparing relations (3.11), (3.13), (3.27) and (3.28), we have:

\[
D_{n}(w) \lesssim n^{(2+\gamma_1)(2-\nu_1)-1} + n^{(2+\gamma_2)(2-\nu_2)-1},
\]

where \( \gamma_1 \) and \( \gamma_2 \) defined as in (3.27) and (3.28).

Now, from (3.7), (3.8), (3.9) and (3.29), for any \( z \in L_R \), we get:

\[
|P_{n}(z)| \lesssim \left[ n^{(2+\gamma_1)(2-\nu_1)} + n^{(2+\gamma_2)(2-\nu_2)} \right] \|P_{n}\|_p
\]

Since this estimate holds for any \( z \in L_R \), then it is also true for \( z \in \overline{G} \). Therefore, we complete the proof of theorem.

\[\square\]

3.2 Proof of Theorem 1.2

Proof. Suppose that \( G \in \text{PAC}(\nu_1,\nu_2) \) for some \( 0 < \nu_1, \nu_2 < 1 \) and \( h(z) \) is defined as in (1.1). For each \( R > 1 \), let \( w = \varphi_R(z) \) denote a univalent conformal mapping \( G_R \) onto the \( B \), normalized by \( \varphi_R(0) = 0, \varphi'_R(0) > 0 \), and let \( \{\zeta_j\} \), \( 1 \leq j \leq m \leq n \), be a zeros of \( P_{n}(z) \) (if any exist) lying on \( G_R \). Let

\[
b_{m,R}(z) := \prod_{j=1}^{m} \zeta_{j,R}(z) = \prod_{j=1}^{m} \frac{\varphi_R(z) - \varphi_R(\zeta_j)}{1 - \varphi_R(\zeta_j) \varphi_R(z)},
\]

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denote a Blaschke function with respect to zeros \( \{\zeta_j\}, 1 \leq j \leq m \leq n \) of \( P_n(z) \) (33). Clearly,

\[
|b_{m,R}(z)| \equiv 1, \ z \in L_R, \text{ and } |b_{m,R}(z)| < 1, \ z \in G_R. \tag{3.31}
\]

For any \( p > 0 \) and \( z \in G_R \), let us set

\[
T_{n,p}(z) := \left[ \frac{P_n(z)}{b_{m,R}(z)} \right]^p. \tag{3.32}
\]

The function \( T_{n,p}(z) \) is analytic in \( G_R \), continuous on \( \overline{G_R} \) and does not have zeros in \( G_R \). We take an arbitrary continuous branch of the \( T_{n,p}(z) \) and for this branch we maintain the same designation. Then, the Cauchy integral representation for the \( T_{n,p}(z) \) at the \( z = z_1 \) gives:

\[
T_{n,p}(z_1) = \frac{1}{2\pi i} \int_{L_R} T_{n,p}(\zeta) \frac{d\zeta}{\zeta - z_1}.
\]

Then, according to (3.31), we obtain:

\[
|P_n(z_1)|^{p/2} \leq \frac{|b_{m,R}(z_1)|^{p/2}}{2\pi} \int_{L_R} \left| \frac{P_n(\zeta)}{|b_{m,R}(\zeta)|} \right|^{p/2} \frac{|d\zeta|}{|\zeta - z_1|} \tag{3.33}
\]

Multiplying the numerator and the denominator of the last integrand by \( h^{1/2}(\zeta) \), replacing the variable \( w = \Phi(z) \) and applying the Hölder inequality, we obtain:

\[
\left( \int_{L_R} |P_n(\zeta)|^2 \frac{|d\zeta|}{|\zeta - z_1|} \right)^2 \leq \int_{|t|=R} \frac{h(\Phi(t)) |P_n(\Phi(t))|^p |\Phi'(t)|^2 |dt|}{h(\Phi(t)) |\Phi(t) - \Phi(w_1)|^2} \cdot \frac{|dt|}{h(\Phi(t)) |\Phi(t) - \Phi(w_1)|^2}, \tag{3.34}
\]

where \( f_{n,p}(t) \) has been defined as in (3.8). Since \( R > 1 \) is arbitrary, then (3.34) holds also for \( R = R_1 := 1 + \frac{\varepsilon_1}{n}, 0 < \varepsilon_1 < 1 \). So, we have:

\[
\left( \int_{L_{R_1}} |P_n(\zeta)|^2 \frac{|d\zeta|}{|\zeta - z_1|} \right)^2 \leq \left( \int_{|t|=R_1} |f_{n,p}(t)|^p |dt| \right) \cdot \left( \int_{|t|=R_1} \frac{|dt|}{h(\Phi(t)) |\Phi(t) - \Phi(w_1)|^2} \right)
\]

and, \( A_n \) and \( D_n(w_j) \) have been defined as in (3.8) for \( R = R_1 \). Therefore, from (3.33) and (3.35), we have:

\[
|P_n(z_1)| \lesssim A_n \cdot D_n(w_1), \tag{3.36}
\]

where, according to (3.9), the estimate

\[
A_n \lesssim n \cdot \|P_n\|_p
\]
is satisfied. For the estimate of the quantity $D_n(w_1)$ we use the notations at the estimation of the $D_n(w)$ as in (3.11)-(3.13). Therefore, under these notations, for the $D_n(w_1)$, we get:

$$D_n(w_1) \leq \int_{\Phi(L^1_{R_1})} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{2+\gamma_1}} \tag{3.37}$$

$$\leq \sum_{i=1}^2 \int_{K^i_1(L_{R_1})} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{2+\gamma_1}} = 2 \sum_{i=1}^2 D_{n,i}^1(w_1).$$

So, we need to evaluate the $D_{n,i}^1(w_1)$ for each $i = 1, 2$. We have:

$$D_{n,1}^1(w_1) = \int_{K^1_1(L_{R_1})} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{2+\gamma_1}} \tag{3.38}$$

$$\leq \int_{K^1_1(L_{R_1})} \frac{|dt|}{|t-w_1|^{2+\gamma_1}(2-\nu_1)} \leq \begin{cases} n^{(2+\gamma_1)(2-\nu_1)-1}, & \text{if } (2+\gamma_1)(2-\nu_1) > 1, \\ \ln n, & \text{if } (2+\gamma_1)(2-\nu_1) = 1, \\ 1, & \text{if } (2+\gamma_1)(2-\nu_1) < 1, \end{cases}$$

and

$$D_{n,2}^2(w_1) = \int_{K^2_1(L_{R_1})} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{2+\gamma_1}} \leq \int_{K^2_1(L_{R_1})} \frac{|dt|}{|t-w_1|^{2+\gamma_1}+\varepsilon} \leq n^{(2+\gamma_1)(1+\varepsilon)-1}. \tag{3.39}$$

Combining relations (3.37) - (3.39), we have:

$$D_n(w_1) \leq \begin{cases} n^{(2+\gamma_1)(2-\nu_1)-1+\varepsilon}, & \text{if } (2+\gamma_1)(2-\nu_1) > 1 - \varepsilon, \\ \ln n, & \text{if } (2+\gamma_1)(2-\nu_1) = 1 - \varepsilon, \\ 1, & \text{if } (2+\gamma_1)(2-\nu_1) < 1 - \varepsilon, \end{cases} \tag{3.40}$$

From the estimations (3.36) and (3.40), we obtain:

$$|P_n(z_1)| \leq \begin{cases} n^{(2+\gamma_1)(2-\nu_1)+\varepsilon}, & \text{if } (2+\gamma_1)(2-\nu_1) > 1 - \varepsilon, \\ (n \ln n)^{\frac{1}{p}}, & \text{if } (2+\gamma_1)(2-\nu_1) = 1 - \varepsilon, \\ n^{\frac{1}{p}}, & \text{if } (2+\gamma_1)(2-\nu_1) < 1 - \varepsilon, \end{cases} ||P_n||_p,$$

and we complete the proof of theorem. \(\square\)

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References