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Zero Sets of Solutions of the Generalized Darboux Equation

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Abstract A non-Euclidean analog of the generalized Darboux equation is considered. For the case where its solutions are radial functions of the second variable we obtain an uniqueness result (Theorem 1), which deals with zero sets of these solutions. The example of the function in Theorem 2 of the paper shows that Theorem 1 cannot be essentially reinforced.

Keywords: Darboux equation, hyperbolic space, transmutation homeomorphisms.

1 Introduction and Statement of Main Results

Let $\mathcal{L}$ be the Laplace-Beltrami operator on a Riemannian manifold X (see, for instance, [1, Ch. 2]). The partial differential equation

$$\mathcal{L}_x\big(f(x,y)\big) = \mathcal{L}_y\big(f(x,y)\big) \tag{1}$$

with $f = f(x,y) \in C^2(X \times X)$ is called the generalizing Darboux equation. Such equations are of considerable interest in their own right, but they are also important for many applications in geometric analysis (see [1], [2]) and integral geometry (see [3]–[5]). In particular, equations of type (1) are closely connected with the mean value operators on symmetric spaces (see [1]–[3], [5]).

In this paper, we investigate zero sets of solutions of the generalized Darboux equation for the case where X is the real hyperbolic space.

We take X as the ball $B = \{x \in \mathbb{R}^n : |x| < 1\}$ with the Riemannian structure

$$\mathrm{d}s^2 = \frac{\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2}{(1-|x|^2)^2}. \tag{2}$$

The Laplace-Beltrami operator for (2) is given by

$$L = \big(1-|x|^2\big)^n \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\big(1-|x|^2\big)^{2-n}\frac{\partial}{\partial x_j}\right).$$

Thus equation (1) has the form

$$\big(1-|x|^2\big)^n \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\big(1-|x|^2\big)^{2-n}\frac{\partial}{\partial x_j}\right) f =$$

$$= \big(1-|y|^2\big)^n \sum_{j=1}^{n} \frac{\partial}{\partial y_j}\left(\big(1-|y|^2\big)^{2-n}\frac{\partial}{\partial y_j}\right) f, \tag{3}$$

where $f = f(x,y) \in C^2(B \times B)$.

For $R \in (0,1)$ and $r \in [0,R)$ we set

$$M_{r,R,1} = \{(x,y) \in B \times B : r \le |x| \le R,\ |y| \le \operatorname{th}(\operatorname{arth}|x| - \operatorname{arth} r)\},$$

$$M_{r,R,2} = \{(x,y) \in B \times B : R \le |x| \le \operatorname{th}(2\operatorname{arth}R - \operatorname{arth} r),$$
$$|y| \le \operatorname{th}(2\operatorname{arth}R - \operatorname{arth}r - \operatorname{arth}|x|)\}.$$

Let $SO(n)$ be the rotation group of $\mathbb{R}^n$.

The main results of this paper are as follow.

Theorem 1 *Let $f \in C^2(B \times B)$ satisfy (3). Suppose that $R \in (0,1)$ and $r \in [0,R)$ are given, and the following conditions hold.*
(i) $f(x,y) = f(x,ky)$ *for all* $x, y \in B,\ k \in SO(n)$.
(ii) $f(x,0) = 0$ *if* $r \le |x| \le R$.
(iii) $f(x,y) = 0$ *for all* $x, y \in B,\ |x| = R$.
Then $f = 0$ in $M_{r,R,1} \cup M_{r,R,2}$. Moreover, if $r = 0$ then $f = 0$ in B.

We need to say a word about condition (i). It is a well-known fact that if $f(x,y)$ is a radial function of y and

$$f(x,y) = h(x,t), \qquad t = \operatorname{arth}|y|,$$

then equation (3) can be rewritten as

$$\left(1-|x|^2\right)^n \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\left(1-|x|^2\right)^{2-n}\frac{\partial}{\partial x_j}\right) h = \frac{\partial^2 h}{\partial t^2} + 2(n-1)\operatorname{cth} 2t\,\frac{\partial h}{\partial t}.$$

This relation is a hyperbolic analog of the Darboux equation. Some Euclidean analogs of Theorem 1 can be found in [4] and [6].

The following result shows that Theorem 1 cannot be essentially improved.

Theorem 2 *Suppose that $R \in (0,1)$ and $\varepsilon \in (0,R)$ are given. Then there exists a nonzero solution $f \in C^2(B \times B)$ of equation (3) such that*

$$f(x,0) = 0 \qquad in \qquad \{x \in B : |x| \le R - \varepsilon\} \tag{4}$$

and conditions (i) *and* (iii) *in Theorem 1 hold.*

For more results on the theory of differential equations on symmetric spaces and their applications, see [2].

2 Basic Notation

In the paper, we use the following standard notations: $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Z}_+$ denote the sets of real, natural, integer and non-negative integers, respectively; Γ is the gamma-function; $F(a,b;c;z)$ is the Gauss hypergeometric function.

The Möbius group $\mathcal{M}(B)$ acts transitively on B by conformal mappings (see, for example, [4, Part 2, Ch. 2]). The Möbius transformations are motions in Poincaré's model of the real hyperbolic space realized on the ball B. The hyperbolic metric d on this space is defined by the equality

$$d(0,x) = \frac{1}{2}\ln\frac{1+|x|}{1-|x|}, \qquad x \in B, \tag{5}$$

and the condition of invariance under the group $\mathcal{M}(B)$. Relation (5) shows that

$$|x| = \operatorname{th} d(0,x), \qquad x \in B.$$

The Riemannian measure $\mathrm{d}\mu$ on B has the form

$$\mathrm{d}\mu(x) = \frac{\mathrm{d}x}{(1-|x|^2)^n}.$$

We recall that $\mathrm{d}\mu$ is invariant under $\mathcal{M}(B)$.

For $R > 0$, we denote by the symbol $B_R(y)$ an open ball with radius R centered at $y \in B$, i.e.,

$$B_R(y) = \{x \in B : d(x,y) < R\}.$$

We set $B_R = B_R(0)$ and $S_R(y) = \{x \in B : d(x,y) = R\}$. Furthermore, let χ_R be the characteristic function (the indicator) of the ball B_R.

We need the following classes of functions and distributions on B: $L(B)$ and $L^{\text{loc}}(B)$ are the classes of functions integrable and locally integrable on B with respect to the measure $\mathrm{d}\mu$; $\mathcal{D}'(B)$ and $\mathcal{E}'(B)$ are the spaces of distributions and compactly supported distributions on B, respectively; $\mathcal{D}(B)$ is the space of compactly supported functions infinite differentiable in B.

Let T be a distribution with compact support in $\mathbb{R}$. Its Fourier transform is defined by the relation

$$\widehat{T}(z) = \langle T, e^{-izt}\rangle, \qquad z \in \mathbb{C}.$$

For a distribution f, $\overline{f}$ denotes its complex conjugation, $\operatorname{supp} f$ stands for the support of f. The symbol $\times$ denotes the convolution of distributions on B in the cases where it exists (see [1, Ch. 2, § 5]). For the convolution of distributions on $\mathbb{R}$, we use the usual symbol "$*$".

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, let ω_{n-1} be the area of the sphere S^{n-1}, let ρ and σ be the polar coordinates of the point $x \in \mathbb{R}^n$ ($\rho = |x|$. If $x \neq 0$, then $\sigma = x/\rho \in S^{n-1}$).

Let $\mathcal{H}_k$ be the space of spherical harmonics of degree k on $\mathbb{S}^{n-1}$, regarded as a subspace of $L^2(\mathbb{S}^{n-1})$ (see [4, Part 1, Ch. 5]), let a_k be the dimension of $\mathcal{H}_k$, and let $\{Y_j^{(k)}\}$ $1 \le j \le a_k$, be an orthonormal basis in $\mathcal{H}_k$. To every function $f \in L^{\text{loc}}(B_R)$ we assign its Fourier series

$$f(x) \sim \sum_{k=0}^{\infty}\sum_{j=1}^{a_k} f_{k,j}(\rho) Y_j^{(k)}(\sigma), \qquad 0 < \rho < \operatorname{th} R,$$

where

$$f_{k,j}(\rho) = \int_{S^{n-1}} f(\rho\sigma)\,\overline{Y_j^{(k)}(\sigma)}\,\mathrm{d}\sigma.$$

We set

$$f^{k,j}(x) = f_{k,j}(\rho)\,Y_j^{(k)}(\sigma).$$

Let $O(n)$ be the orthogonal group of $\mathbb{R}^n$ with the normalized Haar measure $d\tau$, let $T^k(\tau)$ be the restriction of the quasi-regular representation of $O(n)$ to the space $\mathcal{H}_k$ (see [7, Part 2, Ch. 9]), let $\{t_{j,p}^k\}$ $(1 \le j, p \le a_k)$ be the matrix of the representation $T^k(\tau)$, that is

$$\big(T^k(\tau) Y_j^{(k)}\big)(\sigma) = Y_j^{(k)}(\tau^{-1}\sigma) = \sum_{p=1}^{a_k} t_{j,p}^k(\tau)\,Y_p^{(k)}(\sigma)$$

for any $\tau \in O(n)$ and $\sigma \in \mathbb{S}^{n-1}$. Then one has

$$f^{k,j}(x) = a_k \int_{O(n)} f(\tau^{-1}x)\,\overline{t_{j,j}^k(\tau)}\,\mathrm{d}\tau \tag{6}$$

(ñì. [7, Part 2, Ch. 9, formula (9.5)]). Next, for each $f \in \mathcal{D}'(B_R)$ we define the distribution $f^{k,j} \in \mathcal{D}'(B_R)$ by the formula

$$\langle f^{k,j}, g\rangle = \left\langle f, a_k \int_{O(n)} g(\tau^{-1}x)\,t_{j,j}^k(\tau)\,\mathrm{d}\tau \right\rangle, \qquad g \in \mathcal{D}(B_R).$$

For a set $\mathfrak{M}(B_R) \subset \mathcal{D}'(B_R)$ let

$$\mathfrak{M}_{k,j}(B_R) = \{f \in \mathfrak{M}(B_R) : f = f^{k,j}\}, \qquad \mathfrak{M}_{\natural}(B_R) = \mathfrak{M}_{0,1}(B_R).$$

3 The Functions $\boldsymbol{\Phi_{\lambda,k,j}}$

For the rest of the paper, $\lambda \in \mathbb{C}$ and

$$\nu = \nu(\lambda) = \frac{1}{2}\,(i\lambda + n - 1).$$

For $k \in \mathbb{Z}_+$, $j \in \{1, \dots, a_k\}$ and $x \in B\backslash\{0\}$ we put

$$\Phi_{\lambda,k,j}(x) = \Phi_{\lambda,k}(\rho) Y_j^{(k)}(\sigma),$$

where

$$\Phi_{\lambda,k}(\rho) = \frac{\Gamma(\nu+k)\,\Gamma\left(\frac{n}{2}\right)}{\Gamma(\nu)\,\Gamma\left(\frac{n}{2}+k\right)}\,\rho^k\left(1-\rho^2\right)^\nu F\left(\nu+k, \nu+1-\frac{n}{2}; \frac{n}{2}+k; \rho^2\right). \tag{7}$$

For any $m \in \mathbb{Z}$ we consider the differential operator d_m defined on $C^1(0,1)$ as follows:

$$\left(d_m f\right)(t) = \frac{t^m}{(1-t^2)^{m-1}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{1}{t}-t\right)^m f(t)\right), \quad f \in C^1(0,1).$$

Let $\mathcal{L}_k = \mathcal{L} - 4(k-1)(n+k-2)I$, where I is the identity operator. A simple calculation shows that

$$\left(\mathcal{L}_k f\right)(x) = (d_{k-1}d_{2-k-n}u)(\rho) Y_j^{(k)}(\sigma) \tag{8}$$

if $f \in C^2(B_R)$ has the form $f(x) = u(\rho) Y_j^{(k)}(\sigma)$.

Using (7) and [8, formulae 2.8 (25), 2.8 (26))] we easily obtain

$$\left(d_k \Phi_{\lambda,k}\right)(\rho) = (i\lambda - 2k - n + 1)\Phi_{\lambda,k+1}(\rho), \tag{9}$$

$$\left(d_{1-k-n}\Phi_{\lambda,k+1}\right)(\rho) = (i\lambda + 2k + n - 1)\Phi_{\lambda,k}(\rho). \tag{10}$$

In what follows we assume that all functions that are defined and continuous in a punctured neighbourhood of zero in $\mathbb{R}^n$ and admit continuous extension to 0 are defined at 0 by continuity. The functions $\Phi_{\lambda,k,j}$ admit continuous extension to the point $x = 0$, becoming real-analytic functions on B. Formulae (8), (9) and (10) imply that

$$\left(\mathcal{L} + (\lambda^2 + (n-1)^2)I\right)\left(\Phi_{\lambda,k,j}\right) = 0. \tag{11}$$

In addition, the equality

$$\Phi_{\lambda,k,j}(x) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \left(\frac{1-|x|^2}{|x-\eta|^2}\right)^\nu Y_j^{(k)}(\eta)\,\mathrm{d}\eta \tag{12}$$

holds for all $\lambda \in \mathbb{C}$ and $x \in B$ (see [4, Part 2, Ch. 2, formula (2.9)]). Since

$$\frac{1-|x|^2}{|x-\eta|^2} \le \frac{1+|x|}{1-|x|}, \qquad x \in B,\ \eta \in S^{n-1},$$

it follows from (12) that

$$\max_{x\in B_r}\left|\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}\Phi_{\lambda,k,j}(x)\right| = O\left((1+|\lambda|)^{\alpha_1+\cdots+\alpha_n}\, e^{r|\mathrm{Im}\,\lambda|}\right) \tag{13}$$

for $r \in (0,1)$ and $\alpha_1, \dots, \alpha_n \in \mathbb{Z}_+$, where the constant in O does not depend on λ.

Lemma 1 *Let $\lambda, \mu \in \mathbb{C}$, $R \in (0,1)$. Then*

$$\begin{aligned}(\lambda^2-\mu^2)\int_0^R \frac{t^{n-1}}{(1-t^2)^n}\,\Phi_{\lambda,k}(t)\,\Phi_{\mu,k}(t)\,\mathrm{d}t &= \\ = \frac{R^{n-1}}{(1-R^2)^{n-1}}&\left(\Phi_{\lambda,k}(R)\Phi'_{\mu,k}(R) - \Phi_{\mu,k}(R)\Phi'_{\lambda,k}(R)\right).\end{aligned} \tag{14}$$

Proof. Putting $\alpha = \frac{n-2}{2} + k$, we get

$$\Phi_{\lambda,k}(t) = \frac{\Gamma(\nu+k)\,\Gamma\left(\frac{n}{2}\right)}{\Gamma(\nu)\,\Gamma\left(\frac{n}{2}+k\right)}\, t^k\,(1-t^2)^{-k}\,\varphi_{\lambda,\alpha}(\operatorname{arth} t), \tag{15}$$

where

$$\varphi_{\lambda,\alpha}(\xi) = F\left(\alpha + \frac{1+i\lambda}{2},\ \alpha + \frac{1-i\lambda}{2};\ \alpha+1;\ -\operatorname{sh}^2 t\right) \tag{16}$$

(see (7) and [8, Ch. 2.9]). Using now [7, formulae (7.18) and (7.46)] one has

$$(\lambda^2-\mu^2)\int_0^t \varDelta_\alpha(\xi)\,\varphi_{\mu,\alpha}(\xi)\,\varphi_{\lambda,\alpha}(\xi)\,d\xi = \\ = \varDelta_\alpha(t)\left(\varphi_{\lambda,\alpha}(t)\varphi'_{\mu,\alpha}(t) - \varphi_{\mu,\alpha}(t)\varphi'_{\lambda,\alpha}(t)\right),$$

where

$$\varDelta_\alpha(t) = \left(\frac{\sin 2it}{2i}\right)^{2\alpha+1}.$$

This together with (15) implies (14).

Equality (15) implies that for all $k \in \mathbb{Z}_+$, $R \in (0,1)$ the function $\Phi_{\lambda,k}(R)$ is an even entire function of λ. Using [7, Proposition 7.4] we see from Hadamard's theorem [9, Ch. 1, Theorem 13] that $\Phi_{\lambda,k}(R)$ has infinitely many zeros.

Lemma 2 *All the zeros of $\Phi_{\lambda,k}(R)$ are real, simple, and the set of these zeros is symmetric with respect to $\lambda = 0$. In addition $\Phi_{\lambda,k}(R) > 0$ for $i\lambda \in \mathbb{R}$.*

Proof. It follows from (15), (16), and the expansion of F in a hypergeometric series (see [8, Ch. 2, § 2.1, formula (1)]) that $\Phi_{\lambda,k}(R) > 0$ for $i\lambda \in \mathbb{R}$. Next, let $\Phi_{\lambda,k}(R) = 0$ for some $\lambda \in \mathbb{C}$. We claim that $\lambda \in \mathbb{R}$ and $\frac{d}{dt}\,\Phi_{t,k}(R)\Big|_{t=\lambda} \neq 0$. Assume that $\lambda \notin \mathbb{R}$; then $\lambda^2 \neq \overline{\lambda}^2$, since $i\lambda \notin \mathbb{R}$. Putting $\mu = \overline{\lambda}$ in (14) and taking into account that $\Phi_{\overline{\lambda},k}(R) = 0$, we infer that

$$\int_0^R \frac{t^{n-1}}{(1-t^2)^n}\left|\Phi_{\lambda,k}(t)\right|^2 \mathrm{d}t = 0, \tag{17}$$

which is impossible. Now assume that $\frac{d}{dt}\,\Phi_{t,k}(R)\Big|_{t=\lambda} = 0$. Letting $\mu \to \lambda$ in (14) we obtain (17) once again. Hence, all the zeros of $\Phi_{\lambda,k}(R)$ are real and simple. Since the function $\Phi_{\lambda,k}(R)$ is even, this completes the proof of the lemma.

Let $N_k(R)$ be the set of positive zeros λ of the function $\Phi_{\lambda,k}(R)$. Lemma 2 shows that $N_k(R)$ has the form $N_k(r) = \{\lambda_1, \lambda_2, \ldots\}$, where $\lambda_m = \lambda_m(R,k)$ is the sequence of all positive zeros of $\Phi_{\lambda,k}(R)$ numbered in the ascending order. Owing to [9, Ch. 1, Theorem 6], we have

$$\sum_{m=1}^{\infty} \lambda_m^{-1-\varepsilon} < \infty$$

for any $\varepsilon > 0$.

Lemma 3 *Let $\lambda \in N_k(R)$ and*

$$I(\lambda) = \int_0^R \frac{t^{n-1}}{(1-t^2)^n}\,|\Phi_{\lambda,k}(t)|^2\,\mathrm{d}t.$$

Then $I(\lambda) > C\lambda^{-n-2}$, where $C > 0$ does not depend on λ.

Proof. Because of (16), for $t > 0$ one has

$$\varphi_{\lambda,\alpha}(t) = \frac{\Gamma(\alpha+1)\,(\mathrm{sh}2t)^{2\alpha}}{\Gamma(\alpha+\frac{1}{2})\,2^{\alpha-\frac{3}{2}}}\cdot$$

$$\cdot\int_0^t (\mathrm{ch}\,2t - \mathrm{ch}\,2\xi)^{\alpha-\frac{1}{2}} F\left(2\alpha, 0; \alpha+\frac{1}{2}; \frac{\mathrm{ch}\,t - \mathrm{ch}\,\xi}{2\,\mathrm{ch}\,t}\right)\cos\lambda\xi\, d\xi$$

(see [10, equality (2.21)]). Using now (15) and repeating the arguments in [4, Part 2, the proof of Lemma 2.7] we arrive at the desired statement.

Formula (11) with $k = 0$ implies that $\Phi_{\lambda,0}(|x|)$ coincides with the elementary spherical function φ_λ on B (see [8, Ch. 4, §4.2]). The spherical transform $\widetilde{f}(\lambda)$ of a distribution $f \in \mathcal{E}'_\natural(B)$ is defined by

$$\widetilde{f}(\lambda) = \langle f, \varphi_\lambda\rangle. \tag{18}$$

By (18) and (11) we conclude that

$$\widetilde{L^m f}(\lambda) = (-1)^m(\lambda^2 + (n-1)^2)^m \widetilde{f}(\lambda), \qquad m \in \mathbb{Z}_+.$$

This together with (13) shows that for $f \in (\mathcal{E}'_\natural \cap C^{2m})(B)$

$$\widetilde{f}(\lambda) = O\big(|\lambda|^{-2m}\big), \qquad \lambda \to \infty,\ \lambda \in \mathbb{R}, \tag{19}$$

where the constant of the symbol O is independent of λ.

Lemma 4 *Let $T \in \mathcal{E}'_\natural(B)$, $f \in C^2(B)$ and $Lf = -(\lambda^2 + (n-1)^2)f$. Then*

$$(f \times T)(x) = \widetilde{T}(\lambda) f(x), \qquad x \in B. \tag{20}$$

In particular,

$$(\Phi_{\lambda,k,j} \times \chi_r)(x) = \omega_{n-1}\left(\frac{\mathrm{sh}2r}{2}\right)^{n-1}\frac{\Phi_{\lambda,1}(\mathrm{th}r)}{\nu}\,\Phi_{\lambda,k,j}(x) \tag{21}$$

for any $r > 0$.

Proof. The first equality follows from the mean value theorem for the eigenfunctions of the operator L (see [8, Ch. 4, § 2.2]). Next one has

$$\chi_r(\lambda) = \int_{B_r}\Phi_{\lambda,0}(|x|)\mathrm{d}\mu(x) = \omega_{n-1}\int_0^{\mathrm{th}r}\frac{\rho^{n-1}\Phi_{\lambda,0}(\rho)}{(1-\rho^2)^n}\mathrm{d}\rho.$$

Combining this with (10), we obtain

$$\widetilde{\chi}_r(\lambda) = \omega_{n-1}\left(\frac{\mathrm{sh}2r}{2}\right)^{n-1}\frac{\Phi_{\lambda,1}(\mathrm{th}r)}{\nu}.$$

Thus the second equality in the lemma follows from the first with $T = \chi_r$.

4 Linear Homeomorphisms $\mathfrak{A}_{k,j}$

In this section we define an operator allowing the reduction of several problems for convolution in B to the one-dimensional case.

For $f \in L^{\mathrm{loc}}(B)$, $\zeta \in B \setminus \{0\}$, $x \in B \setminus \{0\}$ we set

$$K_\zeta(f)(x) = \frac{1}{\omega_{n-1}}\left(\frac{1-|x|^2}{|x|}\right)^{n-1}(f \times \chi_{\mathrm{arth}\,|x|})(\zeta)Y_1^{(1)}\left(\frac{x}{|x|}\right). \tag{22}$$

Lemma 5 *Let f, $f_n \in L^{\mathrm{loc}}(B)$, $n \in \mathbb{N}$, and assume that $f_n \to f$ in the space $L(E)$ for each compact set $E \subset B$. Then for each $\zeta \in B \setminus \{0\}$ the sequence $\{K_\zeta(f_n)\}$ converges to $K_\zeta(f)$ in $\mathcal{D}'(B)$.*

Proof. Let $\varphi \in \mathcal{D}(B)$, $\operatorname{supp}\varphi \subset B_a$ and let

$$\psi(x) = \frac{1}{\omega_{n-1}} \left(\frac{1-|x|^2}{|x|}\right)^{n-1} \varphi(x) Y_1^{(1)}\left(\frac{x}{|x|}\right).$$

Using (22) we obtain

$$\left|\langle K_\zeta(f_n), \varphi\rangle - \langle K_\zeta(f), \varphi\rangle\right| \leq$$
$$\leq \sup_{x\in B_a} \int_{B_{d(0,x)}(\zeta)} |f_n - f| \mathrm{d}\mu \int_{B_a} |\psi| \mathrm{d}\mu \leq$$
$$\leq \int_{B_a(\zeta)} |f_n - f| \mathrm{d}\mu \int_{B_a} |\psi| \mathrm{d}\mu.$$

Since $f_n \to f$ in $L(B_a(\zeta))$ this implies the desired result.

Lemma 6 *Let $t > 0$ and $k \in \mathbb{Z}_+$. Then there exists $\mathfrak{S}_{t,k} \in \mathcal{E}'(\mathbb{R})$ such that $\operatorname{supp}\mathfrak{S}_{t,k} \subset [-t,t]$ and*

$$\widehat{\mathfrak{S}_{t,k}}(\lambda) = \sqrt{\omega_{n-1}}\, \frac{\Gamma(\nu)}{\Gamma(\nu+k)}\, \Phi_{\lambda,k}(\operatorname{th} t), \qquad \lambda \in \mathbb{C}. \tag{23}$$

Proof. As already pointed out in § 3, for each $t > 0$ the function $\frac{\Gamma(\nu)}{\Gamma(\nu+k)}\, \Phi_{\lambda,k}(\operatorname{th} t)$ is an even entire function of λ. Moreover, it follows from (13) that

$$|\Phi_{\lambda,k}(\operatorname{th} t)| \leq c\, e^{t|\operatorname{Im}\lambda|}, \tag{24}$$

where the constant $c > 0$ does not depend on λ. Now the Paley-Wiener theorem (see [11, Theorem 7.3.1]) completes the proof.

Lemma 7 *Let $a > 0$. There exists a linear homeomorphism $\mathfrak{A}_{k,j} : \mathcal{D}'_{k,j}(B_a) \to \mathcal{D}'_\natural(-a,a)$ such that the following assertions hold.*
(i) For each $\lambda \in \mathbb{C}$,

$$\mathfrak{A}_{k,j}(\Phi_{\lambda,k,j})(t) = \frac{\Gamma(\nu+k)\, \Gamma\left(\frac{n}{2}\right)}{\Gamma(\nu)\, \Gamma\left(\frac{n}{2}+k\right)} \cos\lambda t. \tag{25}$$

(ii) If $f \in L^{\mathrm{loc}}_{k,j}(B_a)$, $t \in (0,a)$, and $\zeta \in S_t$, then

$$n \sqrt{\omega_{n-1}}\, \mathfrak{A}_{1,1}\big(K_\zeta(f)\big) = \frac{\Gamma\left(\frac{n}{2}+k\right)}{\Gamma\left(\frac{n}{2}\right)}\, Y_j^{(k)}\left(\frac{\zeta}{|\zeta|}\right) \big(\mathfrak{A}_{k,j}(f) * \mathfrak{S}_{t,k}\big) \tag{26}$$

in $\mathcal{D}'(t-a, a-t)$.

Proof. According to [7, Theorem 10.21], there exists a linear homeomorphism $\mathfrak{A}_{k,j} : \mathcal{D}'_{k,j}(B_a) \to \mathcal{D}'_\natural(-a,a)$ satisfying (25). Let us prove (26). First of all, we note that the set $Lin\,\{\Phi_{\lambda,k,j},\ \lambda \in \mathbb{C}\}$ is a dense subset of $\mathcal{D}'_{k,j}(B_a)$ (see [7, Proposition 9.9]). Therefore, without loss of generality we can assume that $f = \Phi_{\lambda,k,j}$, $\lambda \in \mathbb{C}$ (see Lemma 5). Next one has

$$n \sqrt{\omega_{n-1}}\, \mathfrak{A}_{1,1}\big(K_\zeta(\Phi_{\lambda,k,j})\big)(\xi) = \Phi_{\lambda,k,j}(\zeta) \cos\lambda\xi \tag{27}$$

(see (21) and (22)). On the other hand,

$$\frac{\Gamma\left(\frac{n}{2}+k\right)}{\Gamma\left(\frac{n}{2}\right)} \big(\mathfrak{A}_{k,j}(\Phi_{\lambda,k,j}) * \mathfrak{S}_{t,k}\big)(\xi) = \sqrt{\omega_{n-1}}\, \Phi_{\lambda,k}(\zeta) \cos\lambda\xi \tag{28}$$

because of (25) and (23). Comparing (27) with (28) we arrive at (26).

5 Proof of Theorem 1

We now proceed to the proof of Theorem 1. Let $f \in C^2(B \times B)$ and suppose that this function satisfies conditions (i)–(iii) in Theorem 1. For each $t \in (0, 1)$, Asgeirsson's mean value theorem (see [1, Ch. 2, § 5.6, Theorem 5.28]) yields

$$\int_{S_t(x)} f(\zeta, 0)\,\mathrm{d}\omega(\zeta) = \int_{S_t(0)} f(x, \zeta)\,\mathrm{d}\omega(\zeta), \qquad x \in B,$$

where $\mathrm{d}\omega$ is the surface element on $S_t(x)$. This equality and condition (i) in Theorem 1 show that

$$f(x, y) = \frac{1}{\omega_{n-1}\,(\operatorname{sh} t \operatorname{ch} t)^{n-1}} \int_{S_t(x)} f(\zeta, 0)\,\mathrm{d}\omega(\zeta) \tag{29}$$

for all $x \in B$, $y \in S_t(0)$. Let $R' = \operatorname{arth} R$, $a = 2R' - \operatorname{arth} r$. Now define $u(x) = f(x, 0)$ for $r \le |x| < a$ and $u(x) = 0$ for $|x| < r$. Relation (6) and property (ii) imply that

$$u^{k,j}(x) = 0 \quad \text{in} \quad B_{R'} \tag{30}$$

for all $k \in \mathbb{Z}_+$, $j \in \{1, \ldots, a_k\}$. In addition, by property (iii) and (29),

$$(u^{k,j} \times \chi_t)(\zeta) = 0, \qquad \zeta \in S_{R'}(0),$$

for each $t \in (0, R' - \operatorname{arth} r)$. Hence

$$\mathfrak{A}_{k,j}(u^{k,j}) * \mathfrak{S}_{R',k} = 0 \quad \text{in} \quad (R' - a, a - R')$$

because of Lemma 7. Using now Lemma 3 and [4, Part 3, Theorem 1.3] we see that

$$\mathfrak{A}_{k,j}(u^{k,j})(t) = \sum_{\lambda \in N_k(R)} c_{\lambda,k,j} \cos \lambda t, \tag{31}$$

where $c_{\lambda,k,j} \in \mathbb{C}$,

$$c_{\lambda,k,j} = O(\lambda^\gamma), \qquad \lambda \to \infty \tag{32}$$

for some $\gamma > 0$, and the series in (31) converges in the space $\mathcal{D}'(-a, a)$. Owing to (25), this means that

$$u^{k,j}(x) = \sum_{\lambda \in N_k(R)} c_{\lambda,k,j}\, \Phi_{\lambda,k,j}(x), \tag{33}$$

where the series converges in $\mathcal{D}'(B_a)$. Let $\varphi_\varepsilon \in \mathcal{D}_\natural(B)$ and $\operatorname{supp} \varphi_\varepsilon \subset B_\varepsilon$, $\varepsilon \in (0, a)$. In view of Lemma 4, we conclude from (33) that

$$(u^{k,j} \times \varphi_\varepsilon)(x) = \sum_{\lambda \in N_k(R)} c_{\lambda,k,j}\, \widetilde{\varphi}_\varepsilon(\lambda)\, \Phi_{\lambda,k,j}(x), \qquad x \in B_{a-\varepsilon}. \tag{34}$$

Together with (32), relation (19) yields

$$c_{\lambda,k,j}\, \widetilde{\varphi}_\varepsilon(\lambda) = O\left(\frac{1}{\lambda^b}\right), \qquad \lambda \to \infty$$

for each $b > 0$. Taking (24) into account we see that the series in (34) converges uniformly on compacts. Therefore, we obtain

$$c_{\lambda,k,j}\, \widetilde{\varphi}_\varepsilon(\lambda) = \left(\int_{B_{R'}} |\Phi_{\lambda,k,j}(x)|^2\,\mathrm{d}\mu(x)\right)^{-1} \int_{B_{R'}} (u^{k,j} \times \varphi_\varepsilon)(x)\, \overline{\Phi_{\lambda,k,j}}(x)\,\mathrm{d}\mu(x)$$

(see Lemma 1). Letting $\varepsilon \to 0$, for a suitable sequence $\{\varphi_\varepsilon\}$ one has

$$c_{\lambda,k,j} = \left(\int_{B_{R'}} |\Phi_{\lambda,k,j}(x)|^2\,\mathrm{d}\mu(x)\right)^{-1} \int_{B_{R'}} u^{k,j}(x)\, \overline{\Phi_{\lambda,k,j}}(x)\,\mathrm{d}\mu(x).$$

Hence $c_{\lambda,k,j} = 0$ for all λ, k, j because of (30). Now we know that the functions $u^{k,j}$ and u vanish in B_a. In view of (29) this gives us the assertion of Theorem 1.

6 Proof of Theorem 2

Owing to [4, the proof of Theorem 2.2.6 (5)], there exists a nonzero function $u \in C^\infty(B)$ such that $u(x) = 0$ for $|x| \leq R - \varepsilon$ and

$$u(x) = \sum_{\lambda \in N_1(r)} c_\lambda \Phi_{\lambda,1,1}(x), \tag{35}$$

where $c_\lambda \in \mathbb{C}$ and the series in (35) converges in the space $C^\infty(B)$. For $x \in B$, $y \in B\backslash\{0\}$ we define

$$f(x, y) = \frac{1}{\omega_{n-1}\,(\operatorname{sh} t \operatorname{ch} t)^{n-1}} \int_{S_R(x)} u(\zeta)\,\mathrm{d}\omega(\zeta), \tag{36}$$

where $R = |y|$. In addition, we set $f(x, 0) = u(x)$, $x \in B$. Then $f \in C^2(B \times B)$ and (3) are satisfied because of [1, Ch. 2, Proposition 4.12]. Next, by the definition of u, relation (4) holds. Finally, in view of (36) and (20), conditions (i) and (iii) in Theorem 1 are satisfied for the function f.

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