Certain Analytic Functions with Missing Coefficients

Cai-Mei Yan\(^1\) and Jin-Lin Liu\(^2\)

\(^1\)Information Engineering College, Yangzhou University, Yangzhou, 225002, P.R. China
Email: cmyan@yzu.edu.cn

\(^2\)Department of Mathematics, Yangzhou University, Yangzhou, 225002, P.R. China
Email: jlliu@yzu.edu.cn

Abstract Let \(A_n\) denote the class of functions of the form \(f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k\), which are analytic in the open unit disk \(U = \{ z : |z| < 1 \}\). In this note we shall find \(\max_{|z|=r<1} \Re\{f'(z) + \alpha zf''(z)\}\) under the condition \(f'(z) \prec 1 + Az + Bz\) for \(f \in A_n\).

Keywords: Analytic function, subordination, missing coefficient.

1 Introduction

Throughout our present investigation, we assume that

\[ n \in \mathbb{N}, \ -1 \leq B < 1, \ B < A, \ \alpha > 0 \ \text{and} \ \beta < 1. \quad (1.1) \]

Let \(A_n\) denote the class of functions of the form:

\[ f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad (1.2) \]

which are analytic in the open unit disk \(U = \{ z : |z| < 1 \}\).

For functions \(f\) and \(g\) analytic in \(U\), we say that \(f\) is subordinate to \(g\) and write \(f(z) \prec g(z) \quad (z \in U)\), if there exists an analytic function \(w(z)\) in \(U\) such that

\[ |w(z)| \leq |z| \ \text{and} \ f(z) = g(w(z)) \quad (z \in U). \]

Furthermore, if the function \(g\) is univalent in \(U\), then

\[ f(z) \prec g(z) \quad (z \in U) \iff f(0) = g(0) \ \text{and} \ f(U) \subset g(U). \]

In a recent paper [3], Gao and Zhou considered the following subclass of \(A_1\):

\[ R(\beta, \alpha) = \{ f \in A_1 : \Re\{f'(z) + \alpha zf''(z)\} > \beta \quad (z \in U)\}. \]

Some interesting properties of the class \(R(\beta, \alpha)\) have been given in [1]. For further information of the class \(R(\beta, \alpha)\) one can see the related papers (see, e.g., \([2,3,4,5,6,7,8,9]\)). Inspired by the above works, in this note we shall find

\[ \max_{|z|=r<1} \Re\{f'(z) + \alpha zf''(z)\}, \]

under the condition \(f'(z) \prec \frac{1+Az}{1+Bz}\).

2 Main Results

Theorem 2.1. Let \(f\) belong to the class \(A_n\) and satisfy

\[ f'(z) \prec \frac{1+Az}{1+Bz} \quad (z \in U). \quad (2.1) \]
Then
\[
\text{Re}\{f'(z) + \alpha zf''(z)\} \leq \frac{1 + (A + B + n\alpha(A - B))r^n + ABr^{2n}}{(1 + Br^n)^2} \quad \text{if } M_n(A, B, \alpha, r) \leq 0,
\] (2.2)
or
\[
\text{Re}\{f'(z) + \alpha zf''(z)\} \leq \frac{L^2_K - 4\alpha^2 K_A K_B}{4\alpha(A - B)r^{n-1}(1 - r^2)K_B} \quad \text{if } M_n(A, B, \alpha, r) \geq 0,
\] (2.3)
where
\[
\begin{align*}
K_A &= 1 - A^2r^{2n} + nAr^{n-1}(1 - r^2), \\
K_B &= 1 - B^2r^{2n} + nBr^{n-1}(1 - r^2), \\
L_n &= 2\alpha(1 - Abr^{2n}) + n\alpha(A + B)r^{n-1}(1 - r^2) + (A - B)r^{n-1}(1 - r^2), \\
M_n(A, B, \alpha, r) &= 2\alpha K_B(1 + Ar^n) - L_n(1 + Br^n).
\end{align*}
\] (2.4)
The result is sharp.

**Proof.** Equality in (2.2) occurs for \( z = 0 \). Thus we assume that \( 0 < |z| = r < 1 \). From (2.1) we can write
\[
f'(z) = \frac{1 + Az^n\varphi(z)}{1 + Bz^n\varphi(z)} \quad (z \in U),
\] (2.5)
where \( \varphi(z) \) is analytic and \( |\varphi(z)| \leq 1 \) in \( U \). It follows from (2.5) that
\[
\begin{align*}
f'(z) + \alpha zf''(z) &= f'(z) + \frac{\alpha(A - B)z^n(n\varphi(z) + z\varphi'(z))}{(1 + Bz^n\varphi(z))^2} \\
&= f'(z) + \frac{n\alpha}{A - B}(A - Bf'(z))(f'(z) - 1) + \frac{\alpha(A - B)z^{n+1}\varphi'(z)}{(1 + Bz^n\varphi(z))^2}.
\end{align*}
\] (2.6)
With the help of the Carathéodory inequality:
\[
|\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - r^2},
\]
we obtain
\[
\text{Re}\left\{\frac{z^{n+1}\varphi'(z)}{(1 + Bz^n\varphi(z))^2}\right\} \leq \frac{r^{n+1}(1 - |\varphi(z)|^2)}{(1 - r^2)|1 + Bz^n\varphi(z)|^2} \\
= \frac{r^{2n}|A - Bf'(z)|^2 - |f'(z) - 1|^2}{(A - B)^2r^{n-1}(1 - r^2)}. \tag{2.7}
\]
Put \( f'(z) = u + iv \quad (u, v \in R) \). Then (2.6) and (2.7) provide
\[
\text{Re}\left\{f'(z) + \alpha zf''(z)\right\} \leq \left(1 + n\alpha\frac{A + B}{A - B}\right) u - \frac{n\alpha A}{A - B} \left|\frac{u^2 - v^2}{A - B}\right| \\
+ \frac{r^{2n}((A - Bu)^2 + (Bu)^2) - ((u - 1)^2 + v^2)}{(A - B)r^{n-1}(1 - r^2)} \\
= \left(1 + n\alpha\frac{A + B}{A - B}\right) u - \frac{n\alpha A}{A - B}(A - Bu^2) + \frac{r^{2n}(A - Bu)^2 - (u - 1)^2}{(A - B)r^{n-1}(1 - r^2)} \\
+ \frac{\alpha}{A - B} \left(nB - \frac{1 - B^2r^{2n}}{r^{n-1}(1 - r^2)}\right) v^2. \tag{2.8}
\]
Note that
\[
\frac{1 - B^2r^{2n}}{r^{n-1}(1 - r^2)} \geq \frac{1 - r^{2n}}{r^{n-1}(1 - r^2)} = \frac{1}{r^{n-1}}(1 + r^2 + r^4 + \cdots + r^{2(n-2)} + r^{2(n-1)}) \\
= \frac{1}{2r^{n-1}}[(1 + r^{2(n-1)}) + (r^2 + r^{2(n-2)}) + \cdots + (r^{2(n-1)} + 1)] \\
\geq n \geq nB. \tag{2.9}
\]
Combining (2.8) and (2.9) we get
\[
\Re \{f'(z) + \alpha zf''(z)\} \leq \left(1 + n\alpha \frac{A + B}{A - B}\right) u - \frac{n\alpha}{A - B}(A + Bu^2) + \alpha \frac{r^{2n}(A - Bu)^2 - (u - 1)^2}{(A - B)r^{n-1}(1 - r^2)} = \psi_n(u) \quad \text{(say)}.
\]

(2.10)

It is well known that for $|\xi| \leq \sigma$ ($\sigma < 1$),
\[
\left|\frac{1 + A\xi}{1 + B\xi} - \frac{1 - AB\sigma^2}{1 - B^2\sigma^2}\right| \leq \frac{(A - B)\sigma}{1 - B^2\sigma^2} \quad \text{(2.11)}
\]
and
\[
\frac{1 - A\sigma}{1 - B\sigma} \leq \Re \left\{\frac{1 + A\xi}{1 + B\xi}\right\} \leq \frac{1 + A\sigma}{1 + B\sigma} \quad \text{(2.12)}
\]

Also (2.5) and (2.12) imply that
\[
\frac{1 - Ar^n}{1 - Br^n} \leq \Re \{f'(z)\} \leq \frac{1 + Ar^n}{1 + Br^n}.
\]

Let us now calculate the maximum value of $\psi_n(u)$ on the segment $\left[\frac{1 - Ar^n}{1 - Br^n}, \frac{1 + Ar^n}{1 + Br^n}\right]$. Obviously,
\[
\psi_n'(u) = 1 + n\alpha \frac{A + B}{A - B} - \frac{2n\alpha B}{A - B} u + 2\alpha \frac{(1 - ABr^{2n}) - (1 - B^2r^{2n})u}{(A - B)r^{n-1}(1 - r^2)} = \psi_n''(u) = -\frac{2\alpha}{A - B} \left(nB + \frac{1 - B^2r^{2n}}{r^{n-1}(1 - r^2)}\right) < 0 \quad \text{(see (2.9))}
\]
and $\psi_n'(u) = 0$ if and only if
\[
u = u_n = \frac{2\alpha(1 - ABr^{2n}) + n\alpha(A + B)r^{n-1}(1 - r^2) + (A - B)r^{n-1}(1 - r^2)}{2\alpha(1 - B^2r^{2n} + nBr^{n-1}(1 - r^2))} = \frac{L_n}{2\alpha K_B} \quad \text{(see (2.4)).}
\]

Since
\[
2\alpha K_B(1 - Ar^n) - L_n(1 - Br^n) = 2\alpha[(1 - Ar^n)(1 - B^2r^{2n}) - (1 - Br^n)(1 - ABr^{2n})] - n\alpha r^{n-1}(1 - r^2)[(A + B)(1 - Br^n) - 2B(1 - Ar^n)] - (A - B)r^{n-1}(1 - r^2)(1 - Br^n) = -2\alpha(A - B)r^n(1 - Br^n) - n\alpha(A - B)r^{n-1}(1 - r^2)(1 + Br^n) - (A - B)r^{n-1}(1 - r^2)(1 - Br^n) < 0,
\]
we see that
\[
u_n > \frac{1 - Ar^n}{1 - Br^n} \quad \text{(2.15)}
\]

But $u_n$ is not always less than $\frac{1 + Ar^n}{1 + Br^n}$. The following two cases arise.

Case (i). $u_n \geq \frac{1 + Ar^n}{1 + Br^n}$, that is, $M_n(A, B, \alpha, r)$ (given by (2.4)) $\leq 0$. In view of $\psi_n'(u_n) = 0$ and (2.13), the function $\psi_n(u)$ is increasing on the segment $\left[\frac{1 - Ar^n}{1 - Br^n}, \frac{1 + Ar^n}{1 + Br^n}\right]$. Therefore we deduce from (2.10) that,
if \( M_n(A, B, \alpha, r) \leq 0 \), then
\[
\text{Re} \{ f'(z) + \alpha zf''(z) \} \leq \psi_n \left( \frac{1 + Ar^n}{1 + Br^n} \right)
\]
\[
= \left( 1 + n\alpha \frac{A + B}{A - B} \right) \left( 1 + Br^n \right) - \frac{n\alpha}{A - B} \left( A + B \left( \frac{1 + Ar^n}{1 + Br^n} \right)^2 \right)
\]
\[
= \frac{1 + Ar^n}{1 + Br^n} - \frac{n\alpha}{A - B} \left( 1 - 1 + Ar^n \right) \left( A - B \left( \frac{1 + Ar^n}{1 + Br^n} \right)^2 \right)
\]
\[
= \frac{1 + (A + B + n\alpha(A - B))r^n + ABr^{2n}}{(1 + Br^n)^2}.
\]
This proves (2.2).

Next we consider the function \( f \) defined by
\[
f(z) = \int_0^z \frac{1 + At^n}{1 + Bt^n} dt
\]
which satisfies the condition (2.1). It is easy to check that
\[
f'(r) + \alpha rf''(r) = \frac{1 + (A + B + n\alpha(A - B))r^n + ABr^{2n}}{(1 + Br^n)^2},
\]
which shows that the inequality (2.2) is sharp.

Case (ii). \( u_n \leq \frac{1 + Ar^n}{1 + Br^n} \), that is, \( M_n(A, B, \alpha, r) \geq 0 \). In this case we easily have
\[
\text{Re} \{ f'(z) + \alpha zf''(z) \} \leq \psi_n(u_n).
\] (2.16)
In view of (2.4), \( \psi_n(u) \) in (2.10) can be written as
\[
\psi_n(u) = \frac{-\alpha KBu^2 + Lu_n - \alpha Ka}{(A - B)r^{n-1}(1 - r^2)}.
\] (2.17)
Therefore, if \( M_n(A, B, \alpha, r) \geq 0 \), then it follows from (2.14), (2.16) and (2.17) that
\[
\text{Re} \{ f'(z) + \alpha zf''(z) \} \leq \frac{-\alpha KBu_n^2 + Lu_n - \alpha Ka}{(A - B)r^{n-1}(1 - r^2)}
\]
\[
= \frac{L^2_n - 4\alpha^2 KA KB}{4\alpha(A - B)r^{n-1}(1 - r^2)K_B}.
\]
To show that the inequality (2.3) is sharp, we take
\[
f(z) = \int_0^z \frac{1 + At^n}{1 + Bt^n} \varphi(t) dt \quad \text{and} \quad \varphi(z) = \frac{z - c_n}{1 - c_n z}
\]
where \( c_n \in R \) is determined by
\[
f'(r) = \frac{1 + Ar^n \varphi(r)}{1 + Br^n \varphi(r)} = u_n \in \left( 1 - \frac{Ar^n}{1 - Br^n} \right) \left( \frac{1 + Ar^n}{1 + Br^n} \right)
\]
Clearly, \(-1 < \varphi(r) \leq 1, -1 \leq c_n < 1, |\varphi(z)| \leq 1 \) (\( z \in U \)), and so \( f \) satisfies the condition (2.1). Since
\[
\varphi'(r) = \frac{1 - c_n^2}{(1 - c_n r)^2} = \frac{1 - |\varphi(r)|^2}{1 - r^2},
\]
from the above argument we find that
\[
f'(r) + \alpha rf''(r) = \psi_n(u_n).
\]
Now the proof of the theorem is completed.
Corollary 2.2. Let $f$ belong to the class $A_1$ and satisfy $\text{Re}\{f'(z)\} > \beta$ ($\beta < 1; z \in U$). Then for $|z| = r < 1$,

$$\text{Re}\{f'(z) + \alpha zf''(z)\} \leq \beta + (1 - \beta) \frac{1 + 2\alpha r - r^2}{(1 - r)^2}.$$  

The result is sharp.

Proof. By considering $\frac{f'(z) - \beta}{1 - n}$ instead of $f'(z)$, we only need to prove the corollary for $\beta = 0$. Setting $n = A = 1$ and $B = -1$ in (2.4), we get

$$K_1 = 2(1 - r^2), \quad K_1 = 0, \quad L_1 = 2\alpha(1 + r^2) + 2(1 - r^2)$$

and

$$M_1(1, -1, \alpha, r) = -2(1 - r)[1 + \alpha - (1 - \alpha)r^2] \leq 0.$$  

Consequently, an application of (2.2) in Theorem 2.1 yields

$$\text{Re}\{f'(z) + \alpha zf''(z)\} \leq \frac{1 + 2\alpha r - r^2}{(1 - r)^2}.$$  

Furthermore the sharpness follows immediately from that of Theorem 2.1.

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References