

Existence of Solutions to Boundary Value Problems for a Class of Nonlinear Fuzzy Fractional Differential Equations

Yupin Wang*, Shurong Sun and Zhenlai Han

School of Mathematical Sciences, University of Jinan, Jinan, Shandong 250022, P R China
Email: wangyupin8252@163.com

Abstract In this paper, we investigate the existence and uniqueness of solution to boundary value problems for a class of nonlinear fuzzy fractional differential equations involving the fuzzy gH -fractional Caputo derivative. By means of the Schauder fixed point theorem in semi-linear spaces and integral inequality technique, some qualitative results of solutions are obtained. An example is provided from which new results are found.

Keywords: Fuzzy fractional differential equations, generalized Hukuhara differentiability, boundary value problems, Schauder fixed point theorem, generalized Gronwall inequality.

1 Introduction

The study of fuzzy differential equations forms a suitable setting for the mathematical modeling of real world problems in which uncertainty or vagueness pervades. As from some concepts of generalized fuzzy derivative begin to emerge in the mid-2000s, this field has developed rapidly and attracted much attention of many scholars. In 2013, Khastan et al. studied the existence of solutions to a class of first-order linear fuzzy differential equations subject to periodic boundary conditions (cf. [1]). In 2016, Wang studied two-point boundary value problems for a class of first-order nonlinear fuzzy differential equations (cf. [2]). Nowadays, the theory of fuzzy differential equations could be applicable in many areas, for instance, physics, thermodynamics, biology, medicine, chemistry and many other fields of science (see e. g., [3,4,5,6,7] and references contained therein).

In 2010, the concept of the solution to fuzzy fractional differential equations (FFDEs) was proposed by Agarwal et al. (cf. [8]). In 2013, the classical Schauder fixed point theorem has been extended to semi-linear Banach spaces and the authors discussed the existence and uniqueness of solution to the following integral equations

$$u(t) = u_0(t) \oplus \mathcal{J}^q f(t, u(t)), \quad 0 < q < 1,$$

which provides a new way to consider FFDEs because semi-linear metric space is a generalization of fuzzy number space (cf. [9]). The establishment of the basic theory of semi-linear metric space constructs a new framework for fuzzy analysis. In 2014, Khastan et al. (cf. [10]) utilized the generalized Schauder fixed point theorem to study the existence of solutions to FFDEs

$$\mathcal{D}^q u(t) = f(t, u(t)), \quad 0 < q < 1.$$

Meanwhile, many experts have made a great number of outstanding contributions in this vigorous and interesting field under generalized Hukuhara differentiability. We can refer to papers [11,12,13,14,15] etc.

Boundary value problems for FFDEs, which are the combination of fuzzy analysis and boundary value problems for fractional differential equations, are a very magical and practical field. However, the works above are curbed in the order $q \in (0, 1]$ as far as we know. In this paper, we initiate to consider the existence and uniqueness of solution to a class of FFDEs

$${}^C_{gH} \mathcal{D}_*^q u(t) = f(t, u(t)), \quad 0 < t < T \tag{1.1}$$

subject to boundary conditions

$$u(0) = \lambda u(T), \quad u'_{gH}(0) = u_0 \in \mathbb{E}_c^1, \tag{1.2}$$

where ${}^C_{gH}\mathcal{D}_*^q$ is the fuzzy gH -fractional Caputo derivative, $q \in (1, 2]$ is a real number, $f : [0, T] \times \mathbb{E}_c^1 \rightarrow \mathbb{E}_c^1 \setminus \mathbb{R}$ is a continuous and compact fuzzy-valued function and $\lambda \in (0, 1) \cup (1, +\infty)$ is a parameter. We utilize Schauder fixed point theorem in semi-linear spaces to study the existence of solutions to boundary value problems (1.1)–(1.2). This work is motivated by papers [2,9,10]. Most of all, this may be the first time to consider such problem about “higher order” fuzzy fractional differential equations.

The remainder of this paper is organized as follows. In Section 2, we collect some basic definitions, properties and lemmas about fuzzy set theory and fractional calculus theory. In Section 3, we investigate the existence and uniqueness of solution to boundary value problems (1.1)–(1.2). In Section 4, an example is given to illustrate our main results. In Section 5, conclusion and some recommendations for future work are drawn.

2 Preliminaries

For convenience, we give some definitions and introduce the necessary notations and lemmas (see e. g., monographs [16,17,18,19] and references contained therein) which will be used in the following sections.

Let \mathbb{E}^1 be the space of all fuzzy sets in \mathbb{R} , that is, \mathbb{E}^1 is the space of all functions $u : \mathbb{R} \rightarrow [0, 1]$ satisfying the following conditions:

- (i) u is normal, i. e., there exists $x_0 \in \mathbb{R}$, such that $u(x_0) = 1$;
- (ii) $[u]^0 = \{x \in \mathbb{R} | u(x) > 0\}$ is compact;
- (iii) u is a convex fuzzy-valued function, i. e., $u(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{u(x_1), u(x_2)\}$ for any $x_1, x_2 \in \mathbb{R}$ and $\lambda \in (0, 1)$;
- (iv) u is an upper semi-continuous function on \mathbb{R} , i. e., $u(x_0) \geq \lim_{x \rightarrow x_0^+} u(x)$ for any $x_0 \in \mathbb{R}$.

Usually, \mathbb{E}^1 is also called the space of fuzzy numbers. And $\mathbb{R} \subseteq \mathbb{E}^1$ if real number set \mathbb{R} is understood as $\mathbb{R} = \{\chi_{\{x\}} | x \text{ is usual real number}\}$.

Define \underline{u} and \bar{u} respectively as the lower and upper branches of the fuzzy set $u \in \mathbb{E}^1$, respectively. The set

$$[u]^\alpha = \{x \in \mathbb{R} | u(x) \geq \alpha\} := [\underline{u}^\alpha, \bar{u}^\alpha]$$

is called the α -level set of the fuzzy set u , where $\alpha \in (0, 1]$. From (i)-(iv), it follows that the α -level set $[u]^\alpha \in K_c(\mathbb{R})$ for all $\alpha \in [0, 1]$, where $K_c(\mathbb{R})$ is the family of all nonempty, compact and convex subsets of \mathbb{R} .

The following operations, which are based on a generalized Zadeh’s extension principle, define a semi-linear structure on \mathbb{E}^1 :

$$\begin{aligned} - (u \oplus v)(x) &= \sup_{x_1+x_2=x} \min\{u(x_1), v(x_2)\}; \\ - (\lambda u)(x) &= \begin{cases} u(x/\lambda), & \lambda \neq 0, \\ \chi_{\{0\}}, & \lambda = 0, \end{cases} \end{aligned}$$

where $u, v \in \mathbb{E}^1$, $\lambda \in \mathbb{R}$. The α -level set of fuzzy sets satisfies the following properties (see [17]):

$$\begin{aligned} - [u \oplus v]^\alpha &= [u]^\alpha + [v]^\alpha = [\underline{u}^\alpha + \underline{v}^\alpha, \bar{u}^\alpha + \bar{v}^\alpha]; \\ - [\lambda u]^\alpha &= \lambda [u]^\alpha = \begin{cases} [\lambda \underline{u}^\alpha, \lambda \bar{u}^\alpha], & \lambda \geq 0, \\ [\lambda \bar{u}^\alpha, \lambda \underline{u}^\alpha], & \lambda < 0. \end{cases} \end{aligned}$$

Here, $[u]^\alpha + [v]^\alpha$ means the usual addition of two subsets of \mathbb{R} and $\lambda [u]^\alpha$ means the usual product between a scalar and a subset of \mathbb{R} .

Lemma 2.1. [20]

- (i) If we denote $\widehat{0} = \chi_{\{0\}}$ then $\widehat{0} \in \mathbb{E}^1$ is neutral element with respect to \oplus , i. e., $u \oplus \widehat{0} = \widehat{0} \oplus u = u$, for all $u \in \mathbb{E}^1$;
- (ii) With respect to $\widehat{0}$, none of $u \in \mathbb{E}^1 \setminus \mathbb{R}$ has inverse in \mathbb{E}^1 (with respect to \oplus);

(iii) For any $a, b \in \mathbb{R}$, with $a, b \geq 0$ or $a, b \leq 0$ and any $u \in \mathbb{E}^1$, we have $(a + b)u = au \oplus bu$. For general $a, b \in \mathbb{R}$, the above property does not hold;

(iv) For any $\lambda \in \mathbb{R}$ and any $u, v \in \mathbb{E}^1$, we have $\lambda(u \oplus v) = \lambda u \oplus \lambda v$;

(v) For any $\lambda, \mu \in \mathbb{R}$ and any $u, v \in \mathbb{E}^1$, we have $\lambda(\mu u) = (\lambda\mu)u$.

Lemma 2.2. Suppose that $\lambda \in \mathbb{R}$, $w \in \mathbb{E}^1 \setminus \mathbb{R}$, then

$$u = \lambda(u \oplus w) \quad (2.1)$$

and

$$v = \lambda v \oplus w \quad (2.2)$$

have solutions in $\mathbb{E}^1 \setminus \mathbb{R}$ if and only if $\lambda \in (-1, 1)$. In addition,

$$u = \begin{cases} \frac{\lambda}{1-\lambda}w, & \lambda \in [0, 1), \\ \frac{\lambda^2}{1-\lambda^2}w \oplus \frac{\lambda}{1-\lambda^2}w, & \lambda \in (-1, 0) \end{cases} \quad \text{and} \quad v = \begin{cases} \frac{1}{1-\lambda}w, & \lambda \in [0, 1), \\ \frac{1}{1-\lambda^2}w \oplus \frac{\lambda}{1-\lambda^2}w, & \lambda \in (-1, 0). \end{cases}$$

Proof. Necessity has been given by [2, Lemma 2]. Here we only discuss the sufficiency. On the one hand, set $u_1 = \frac{\lambda}{1-\lambda}w \in \mathbb{E}^1 \setminus \mathbb{R}$ when $\lambda \in [0, 1)$. From Lemma 2.1, for the right-hand side of fuzzy algebraic equation (2.1), we have

$$\lambda(u_1 \oplus w) = \lambda \left(\frac{\lambda}{1-\lambda}w \oplus w \right) = \frac{\lambda^2}{1-\lambda}w \oplus \lambda w = \frac{\lambda}{1-\lambda}w = u_1,$$

which means u_1 is a solution to equation (2.1) when $\lambda \in [0, 1)$. On the other hand, let $u_2 = \frac{\lambda^2}{1-\lambda^2}w \oplus \frac{\lambda}{1-\lambda^2}w$ when $\lambda \in (-1, 0)$. For the right-hand side of equation (2.1), we have

$$\lambda(u_2 \oplus w) = \frac{\lambda^3}{1-\lambda^2}w \oplus \frac{\lambda^2}{1-\lambda^2}w \oplus \lambda w = \frac{\lambda}{1-\lambda^2}w \oplus \frac{\lambda^2}{1-\lambda^2}w = u_2,$$

which means u_2 is a solution to equation (2.1) when $\lambda \in (-1, 0)$. Therefore, equation (2.1) has solution when $\lambda \in (-1, 1)$. As a similar argument as above, we can discuss fuzzy algebraic equation (2.2). So we omit here. The proof is completed. \square

As a generalization of Hausdorff-Pompeiu metric on compact and convex set, a metric d on \mathbb{E}^1 can be defined by

$$d(u, v) = \sup_{\alpha \in [0, 1]} d_H([u]^\alpha, [v]^\alpha) = \sup_{\alpha \in [0, 1]} \max\{|u^\alpha - v^\alpha|, |\bar{u}^\alpha - \bar{v}^\alpha|\},$$

where $u, v \in \mathbb{E}^1$ and d_H is the Hausdorff-Pompeiu metric.

Definition 2.1. [21] Let $u, v \in \mathbb{E}^1$. If there exists unique fuzzy number $w \in \mathbb{E}^1$ such that $v \oplus w = u$, then w is called the H -difference of u and v , which is denoted $u \ominus v$.

Lemma 2.3. [17, 18] If $u, v, x, y \in \mathbb{E}^1$, then

- (i) $d(u \oplus x, v \oplus x) = d(u, v)$;
- (ii) $d(\lambda u, \lambda v) = |\lambda|d(u, v)$, $\lambda \in \mathbb{R}$;
- (iii) $d(u \oplus x, v \oplus y) \leq d(u, v) + d(x, y)$;
- (iv) $d(\lambda u, \mu u) = |\lambda - \mu|d(u, \bar{0})$, $\lambda, \mu \geq 0$;
- (v) $d(u \ominus x, v \ominus y) \leq d(u, v) + d(x, y)$, provided the differences $u \ominus x$ and $v \ominus y$ exist;
- (vi) (\mathbb{E}^1, d) is a complete metric space.

Denote \mathbb{E}_c^1 as the space of fuzzy number $u \in \mathbb{E}^1$ with the property that the function $\alpha \rightarrow [u]^\alpha$ is continuous in the meaning of the Hausdorff-Pompeiu metric on $[0, 1]$. According to Theorem 3.1 in [22], we know (\mathbb{E}_c^1, d) is a complete metric space. Let $C^{\mathbb{E}}[a, b]$ denote a space of fuzzy-valued functions which are continuous on $[a, b]$. Then $(C^{\mathbb{E}}([a, b], \mathbb{E}_c^1), D)$ is a complete metric space, where

$$D(u, v) := \sup_{t \in [0, 1]} d(u(t), v(t)).$$

Next, we list some definitions and lemmas about relative compactness in \mathbb{E}_c^1 .

Definition 2.2. [22] A subset $A \subseteq \mathbb{E}_c^1$ is said to be compact-supported if there exists a compact set $K \subseteq \mathbb{R}$ such that $[y]^0 \subseteq K$ for all $y \in A$.

Definition 2.3. [22] A subset $A \subseteq \mathbb{E}_c^1$ is said to be level-equicontinuous at $\alpha_0 \in [0, 1]$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|\alpha - \alpha_0| < \delta$ implies $d_H([y]^\alpha, [y]^{\alpha_0}) < \varepsilon$, for all $y \in A$.

A is level-equicontinuous on $[0, 1]$ if A is level-equicontinuous at α for all $\alpha \in [0, 1]$.

Lemma 2.4. (Arzelá-Ascoli Theorem)[22, Theorem 4.1] Let M be a compact metric space, N be a metric space and $E \subseteq C^{\mathbb{E}}(M, N)$. Then E is relatively compact if and only if

- (i) E equicontinuous;
- (ii) For every $a \in M$, the set $E(a) = \{f(a) | f \in E\}$ is relatively compact in N .

Lemma 2.5. [22, Theorem 4.5] Let A be a compact-supported subset of \mathbb{E}_c^1 . Then the following assertions are equivalent:

- (a) A is a relatively compact subset of (\mathbb{E}_c^1, d) ;
- (b) A is level-equicontinuous on $[0, 1]$.

Definition 2.4. [10] A continuous function $f : [0, 1] \times \mathbb{E}_c^1 \rightarrow \mathbb{E}_c^1$ is said to be compact if for every subinterval $I \subseteq [0, 1]$ and every bounded subset $A \subseteq \mathbb{E}_c^1$, then $f(I \times A)$ is relatively compact in (\mathbb{E}_c^1, d) .

Definition 2.5. [17, Definition 2.4.1] The Aumann integral of a fuzzy-valued function $F : I \rightarrow \mathbb{E}^1$ over I is defined levelwise

$$\left[\int_I F(t) dt \right]^\alpha = \int_I F_\alpha(t) dt = \left\{ \int_I g(t) dt : g \in S(F_\alpha) \right\}$$

for all $\alpha \in (0, 1]$, where $S(F_\alpha)$ is the subset of all integrable selections of set-valued mapping F_α .

Lemma 2.6. [23] Let $F, G : I \rightarrow \mathbb{E}^1$ be integrable and $\lambda \in \mathbb{R}$. Then

- (i) $\int_I (F \oplus G) = \int_I F \oplus \int_I G$;
- (ii) $\int_I (\lambda F) = \lambda \int_I F$;
- (iii) $d(F, G)$ is integrable on I ;
- (iv) $d(\int_I F, \int_I G) \leq \int_I d(F, G)$.

Definition 2.6. [10, Definition 2.10] Let $F \in C^{\mathbb{E}}(a, b) \cap L^{\mathbb{E}}(a, b)$. The fuzzy fractional integral of order $q > 0$ of F is defined as

$$\mathfrak{J}^q F(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} F(s) ds, \quad t \in (a, b)$$

provided that the integral in the right-hand side is pointwise well-defined. For $q = 1$ we obtain $\mathfrak{J}^1 F(t) = \int_a^t F(s) ds$, that is, the classical integral operator.

Lemma 2.7. [24] Let $F : [a, b] \rightarrow \mathbb{E}^1$ be an integrable fuzzy function and $p, q > 0$, then

$$\mathfrak{J}^p \mathfrak{J}^q F(t) = \mathfrak{J}^{p+q} F(t).$$

Definition 2.7. [25,26] Given two fuzzy numbers $u, v \in \mathbb{E}^1$, the generalized Hukuhara difference (gH -difference for short) is the fuzzy number w , if it exists, such that

$$u \ominus_{gH} v = w \iff \begin{cases} \text{(i)} & u = v \oplus w, \\ \text{or (ii)} & v = u \oplus (-1)w. \end{cases}$$

It is easy to show that (i) and (ii) are both valid if and only if w is a crisp number.

Definition 2.8. [26] Let $t_0 \in (a, b)$ and h be such that $t_0 + h \in (a, b)$. Then the generalized Hukuhara derivative of a function $F : (a, b) \rightarrow \mathbb{E}^1$ at t_0 is defined as

$$F'_{gH}(t_0) = \lim_{h \rightarrow 0} \frac{1}{h} [F(t_0 + h) \ominus_{gH} F(t_0)]. \tag{2.3}$$

If $F'_{gH}(t_0) \in \mathbb{E}^1$ satisfying (2.3) exists, we say that F is generalized Hukuhara differentiable (gH -differentiable for short) at t_0 .

Remark 2.1. Let $F : [a, b] \rightarrow \mathbb{E}^1$. One says F is twice gH -differentiable at $t_0 \in (a, b)$, if $F'_{gH}(t)$ exists on a neighborhood of t_0 as a fuzzy-valued function and it is gH -differentiable at t_0 . Similarly, we can define higher order gH -derivatives of fuzzy-valued function.

Definition 2.9. [26] Let $F : [a, b] \rightarrow \mathbb{E}^1$ and $t_0 \in (a, b)$ with $\underline{F}^\alpha(t)$ and $\overline{F}^\alpha(t)$ both differentiable at t_0 . We say that F is (i)- gH -differentiable at t_0 if

$$[F'_{gH}(t_0)]^\alpha = [(\underline{F}^\alpha)'(t_0), (\overline{F}^\alpha)'(t_0)], \quad 0 \leq \alpha \leq 1,$$

and F is (ii)- gH -differentiable at t_0 if

$$[F'_{gH}(t_0)]^\alpha = [(\overline{F}^\alpha)'(t_0), (\underline{F}^\alpha)'(t_0)], \quad 0 \leq \alpha \leq 1.$$

Definition 2.10. [11, Definition 3.1] Let $F_{gH}^{(m)} \in C^{\mathbb{E}}[a, b] \cap L^{\mathbb{E}}[a, b]$. The fuzzy gH -fractional Caputo differentiability of fuzzy-valued function F is defined as following:

$$({}^C_{gH}\mathfrak{D}_*^q F)(t) = (\mathfrak{J}^{m-q} F_{gH}^{(m)})(t) = \frac{1}{\Gamma(m-q)} \int_a^t (t-s)^{m-q-1} F_{gH}^{(m)}(s) ds,$$

where $m - 1 < q \leq m$, $m \in \mathbb{N}_+$ and $t \geq a$.

Definition 2.11. [11, Definition 3.4] Let $F : [a, b] \rightarrow \mathbb{E}^1$ be fuzzy gH -fractional Caputo differentiable at $t_0 \in (a, b)$. We say that F is ${}^{CF}[(i)\text{-}gH]$ -differentiable at t_0 if

$$\left[({}^C_{gH}\mathfrak{D}_*^q F)(t_0) \right]^\alpha = [({}^C D_*^q \underline{F}^\alpha)(t_0), ({}^C D_*^q \overline{F}^\alpha)(t_0)], \quad 0 \leq \alpha \leq 1,$$

and F is ${}^{CF}[(ii)\text{-}gH]$ -differentiable at t_0 if

$$\left[({}^C_{gH}\mathfrak{D}_*^q F)(t_0) \right]^\alpha = [({}^C D_*^q \overline{F}^\alpha)(t_0), ({}^C D_*^q \underline{F}^\alpha)(t_0)], \quad 0 \leq \alpha \leq 1,$$

where ${}^C D_*^q$ is the Caputo fractional derivative of a real-valued function.

According to Definition 2.10, we can get the following lemma immediately.

Lemma 2.8. Let F have $[q]$ th-order Caputo gH -derivative and $q > 1$, where $[q]$ denotes the ceiling function of the number q . Then

$$({}^C_{gH}\mathfrak{D}_*^q F)(t) = ({}^C_{gH}\mathfrak{D}_*^{q-1} F'_{gH})(t).$$

Remark 2.2. From the lemma above, it is not difficult to see $({}^C_{gH}\mathfrak{D}_*^q F_{gH}^{(m)})(t) = ({}^C_{gH}\mathfrak{D}_*^{q+m} F)(t)$ if $F_{gH}^{(m)}(t)$ and $({}^C_{gH}\mathfrak{D}_*^{q+m} F)(t)$ exist for $m \in \mathbb{N}$ and $q \in \mathbb{R}_+$. But, it is not always true for the case $({}^C_{gH}\mathfrak{D}_*^{q_1} \cdot {}^C_{gH}\mathfrak{D}_*^{q_2} F)(t) = ({}^C_{gH}\mathfrak{D}_*^{q_1+q_2} F)(t)$ where $q_1, q_2 \in \mathbb{R}_+$, even $({}^C_{gH}\mathfrak{D}_*^{q_1+q_2} F)(t)$ exists.

Lemma 2.9. [11] Let $0 < q \leq 1$ and $t_0 \in \mathbb{R}$, initial value problem

$$\begin{cases} {}^C_{gH}\mathfrak{D}_*^q u(t) = f(t, u(t)), \\ u(t_0) = u_0 \in \mathbb{E}^1 \end{cases}$$

is equivalent to the following integral equation

$$u(t) \ominus_{gH} u(t_0) = \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s, u(s)) ds. \tag{2.4}$$

Remark 2.3. In particular, two cases of existence of the generalized H -difference imply that equation (2.4) is actually a unified formulation of the following integral equations

$$u(t) = u_0 \oplus \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s, u(s)) ds$$

if $u(t)$ is ${}^{CF}[(i)\text{-}gH]$ -differentiable and

$$u(t) = u_0 \ominus \frac{-1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s, u(s)) ds$$

if $u(t)$ is ${}^{CF}[(ii)\text{-}gH]$ -differentiable.

Remark 2.4. Take $q = 1$ in Lemma 2.9. Then we immediately get a similar result in integer order case.

Lemma 2.10. Let $1 < q \leq 2$. Assume that $f : [0, T] \times \mathbb{E}^1 \rightarrow \mathbb{E}^1$ is a continuous function. A fuzzy-valued function $u : [0, T] \rightarrow \mathbb{E}^1$ is a solution to the initial value problem

$$\begin{cases} {}^C_{gH} \mathfrak{D}_*^q u(t) = f(t, u(t)), & (2.5) \\ u(0) = k_1 \in \mathbb{E}^1, & (2.6) \\ u'_{gH}(0) = k_2 \in \mathbb{E}^1 & (2.7) \end{cases}$$

if and only if u satisfies one of the following fuzzy integral equations:

$$u(t) = k_1 \oplus k_2 t \oplus \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds, \tag{2.8}$$

where u is $(i)\text{-}gH$ -differentiable and u'_{gH} is ${}^{CF}[(i)\text{-}gH]$ -differentiable, or

$$u(t) = k_1 \oplus k_2 t \ominus \frac{-1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds,$$

where u is $(ii)\text{-}gH$ -differentiable and u'_{gH} is ${}^{CF}[(ii)\text{-}gH]$ -differentiable, or

$$u(t) = k_1 \ominus (-1) \left[k_2 t \oplus \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds \right],$$

where u is $(ii)\text{-}gH$ -differentiable and u'_{gH} is ${}^{CF}[(i)\text{-}gH]$ -differentiable, or

$$u(t) = k_1 \ominus (-1) \left[k_2 t \ominus \frac{-1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds \right],$$

where u is $(ii)\text{-}gH$ -differentiable and u'_{gH} is ${}^{CF}[(ii)\text{-}gH]$ -differentiable.

Proof. According to Lemma 2.8, equation (2.5) can be rewritten as

$$({}^C_{gH} \mathfrak{D}_*^{q-1} u'_{gH})(t) = f(t, u(t)).$$

Without loss of generality, we only discuss the case that u is $(i)\text{-}gH$ -differentiable and u'_{gH} is ${}^{CF}[(i)\text{-}gH]$ -differentiable. The proofs of the other cases are similar.

Notice continuity of $f : [0, T] \times \mathbb{E}^1 \rightarrow \mathbb{E}^1$ and $u : [0, T] \rightarrow \mathbb{E}^1$. According to Lemma 2.9 and Remark 2.3, equation (2.5) with the condition (2.7) is equivalent to

$$u'_{gH}(t) = k_2 \oplus \mathfrak{J}^{q-1} f(t, u(t)) = k_2 \oplus \frac{1}{\Gamma(q-1)} \int_0^t (t-s)^{q-2} f(s, u(s)) ds. \tag{2.9}$$

From Remark 2.4 and initial condition (2.6), equation (2.9) is equivalent to

$$u(t) = k_1 \oplus k_2 t \oplus \mathfrak{J}^1 \mathfrak{J}^{q-1} f(t, u(t)).$$

According to Lemma 2.7, we get

$$u(t) = k_1 \oplus k_2 t \oplus \mathfrak{J}^q f(t, u(t)),$$

that is, equation (2.8). Conversely, condition $u'_{gH}(0) = k_2$ is assured by equation (2.9). Meanwhile, equation (2.8) apparently satisfies (1.1) as well as condition (1.2). The proof is completed. \square

At the end of this part, we give Schauder fixed point theorem in semi-linear spaces and generalized Gronwall inequality, which are fundamental in the proofs of our main results.

Lemma 2.11. (Schauder fixed point theorem in semi-linear spaces)[9, Theorem 3.4] *Let B be a nonempty, closed, bounded and convex subset of a semi-linear Banach space S having the cancellation property and suppose $P : B \rightarrow B$ is a compact operator. Then P has at least one fixed point in B .*

Lemma 2.12. (Generalized Gronwall inequality)[27, Theorem 1] *Suppose $q > 0$, $a(t)$ is a nonnegative function locally integrable on $0 \leq t < T$ (some $T \leq +\infty$) and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \leq t < T$, $g(t) \leq M$ (constant), and suppose $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with*

$$u(t) \leq a(t) + g(t) \int_0^t (t-s)^{q-1} u(s) ds$$

on this interval. Then

$$u(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(q))^n}{\Gamma(nq)} (t-s)^{nq-1} a(s) \right] ds, \quad 0 \leq t < T.$$

3 Main Results

At the beginning of this section, we give integral representation of boundary value problems (1.1)–(1.2). Before this, we give the definition of solutions to these problems.

Definition 3.1. A function $u \in C^{\mathbb{E}}([0, T], \mathbb{E}_c^1)$ is said to be a $C^F[(m-n)-gH]$ -differentiable solution (m and n can be taken as i or ii) to boundary value problems (1.1)–(1.2), if u is $(m)-gH$ -differentiable and u'_{gH} is $C^F[(n)-gH]$ -differentiable on the entire interval $[0, T]$ as well as u satisfies (1.1)–(1.2).

Lemma 3.1. A fuzzy-valued function $u : [0, T] \rightarrow \mathbb{E}_c^1$ is a solution to the problems (1.1)–(1.2) if and only if u satisfies one of the following fuzzy integral equations:

(K1)

$$u(t) = \Lambda_1 \oplus u_0 t \oplus \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds, \quad t \in [0, T] \tag{3.1}$$

if $\lambda \in (0, 1)$, u is $(i)-gH$ -differentiable and u'_{gH} is $C^F[(i)-gH]$ -differentiable, or
(K2)

$$u(t) = \Lambda_2 \oplus u_0 t \ominus \frac{-1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds, \quad t \in [0, T]$$

if $\lambda \in (0, 1)$, u is $(i)-gH$ -differentiable and u'_{gH} is $C^F[(ii)-gH]$ -differentiable, or
(K3)

$$u(t) = \Lambda_1 \ominus (-1) \left[u_0 t \oplus \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds \right], \quad t \in [0, T] \tag{3.2}$$

if $\lambda \in (1, +\infty)$, u is $(ii)-gH$ -differentiable and u'_{gH} is $C^F[(i)-gH]$ -differentiable, or

(K4)

$$u(t) = \Lambda_2 \ominus (-1) \left[u_0 t \ominus \frac{-1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds \right], \quad t \in [0, T]$$

if $\lambda \in (1, +\infty)$, u is (ii)-gH-differentiable and u'_{gH} is ${}^{CF}[(ii)\text{-gH}]$ -differentiable,

where

$$\Lambda_1 = \frac{\lambda}{1-\lambda} \left(u_0 T \oplus \frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1} f(s, u(s)) ds \right)$$

and

$$\Lambda_2 = \frac{\lambda}{1-\lambda} \left(u_0 T \ominus \frac{-1}{\Gamma(q)} \int_0^T (T-s)^{q-1} f(s, u(s)) ds \right).$$

Proof. Let $u(0) = u_1 \in \mathbb{E}_c^1$. Then, the proof of this lemma is just divided into two cases for the sake of clarity.

Case 1. $\lambda \in (0, 1)$. Firstly, we consider situation (K1). By the Lemma 2.10, with initial conditions $u(0) = u_1$ and $u'_{gH}(0) = u_0$, equation (1.1) is equivalent to the fuzzy integral equation

$$u(t) = u_1 \oplus u_0 t \oplus \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds, \quad t \in [0, T]. \tag{3.3}$$

To meet boundary value condition $u(0) = \lambda u(T)$, equation (3.3) must satisfy

$$\lambda \left(u_1 \oplus u_0 T \oplus \frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1} f(s, u(s)) ds \right) = u_1. \tag{3.4}$$

By Lemma 2.2, equation (3.4) has solution

$$u_1 = \frac{\lambda}{1-\lambda} \left(u_0 T \oplus \frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1} f(s, u(s)) ds \right) = \Lambda_1.$$

By substituting Λ_1 into (3.3), we get (3.1). Conversely, equation (3.1) apparently satisfies (1.1) as well as condition (1.2). For situation (K2), the proof is quite similar to situation (K1) and so is omitted.

Case 2. $\lambda \in (1, +\infty)$. Firstly, we consider situation (K3). By the Lemma 2.10, with initial conditions $u(0) = u_1$ and $u'_{gH}(0) = u_0$, equation (1.1) is equivalent to the fuzzy integral equation

$$u(t) = u_1 \ominus (-1) \left[u_0 t \oplus \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds \right], \quad t \in [0, T]. \tag{3.5}$$

To meet boundary value condition $u(0) = \lambda u(T)$, equation (3.5) must satisfy

$$\lambda \left(u_1 \ominus (-1) \left[u_0 T \oplus \frac{1}{\Gamma(q)} \int_0^T (t-s)^{q-1} f(s, u(s)) ds \right] \right) = u_1,$$

which is equivalent to

$$u_1 = \frac{1}{\lambda} u_1 \oplus (-1) \left[u_0 T \oplus \frac{1}{\Gamma(q)} \int_0^T (t-s)^{q-1} f(s, u(s)) ds \right]. \tag{3.6}$$

By Lemma 2.2, equation (3.6) has solution

$$u_1 = \frac{\lambda}{1-\lambda} \left(u_0 T \oplus \frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1} f(s, u(s)) ds \right) = \Lambda_1.$$

By substituting Λ_1 into (3.5), we get (3.2). Conversely, equation (3.2) apparently satisfies (1.1) as well as condition (1.2). For situation (K4), the proof is quite similar to situation (K3) and so is omitted. The proof is completed. \square

Corollary 3.1. If $\lambda = 0$, boundary value problems (1.1)–(1.2) degenerate into the following initial value problem

$$\begin{cases} {}^C_{gH}\mathfrak{D}_*^q u(t) = f(t, u(t)), \\ u(0) = \widehat{0}, u'_{gH}(0) = u_0 \in \mathbb{E}^1 \end{cases} \tag{3.7}$$

and it is equivalent to

$$u(t) = u_0 t \oplus \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds$$

if u is (i)- gH -differentiable and u'_{gH} is ${}^{CF}[(i)-gH]$ -differentiable, or

$$u(t) = u_0 t \ominus \frac{-1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds$$

if u is (i)- gH -differentiable and u'_{gH} is ${}^{CF}[(ii)-gH]$ -differentiable.

In the following, we impose growth conditions on f which allow us to establish some results for existence and uniqueness of solution to boundary value problems (1.1)–(1.2).

Theorem 3.1. Suppose that $\lambda \in (0, 1)$ and

(H1) $f : [0, T] \times \mathbb{E}_c^1 \rightarrow \mathbb{E}_c^1 \setminus \mathbb{R}$ is a given continuous and compact fuzzy-valued function;

(H2) $M := \sup_{(t,u) \in [0,T] \times \mathbb{E}_c^1} d(f(t, u), \widehat{0}) < +\infty$.

Then boundary value problems (1.1)–(1.2) have at least one ${}^{CF}[(i-i)-gH]$ -differentiable solution or ${}^{CF}[(i-ii)-gH]$ -differentiable solution.

Proof. Let $S = C^{\mathbb{E}}([0, T], \mathbb{E}_c^1)$. Then S is a semi-linear Banach space having the cancellation property with respect to \oplus . Define a set $B \subset S$ by

$$B = \{u \in S \mid D(u, \widehat{0}) \leq R\} \tag{3.8}$$

for some fixed $R > 0$. It is easy to know that B is a nonempty, closed, bounded and convex subset of S . Choose $R = \frac{T}{1-\lambda} d(u_0, \widehat{0}) + \frac{MT^q}{(1-\lambda)\Gamma(q+1)} + 1$. Then, the proof of this theorem is just divided into two cases for the sake of clarity.

Case 1. u is (i)- gH -differentiable and u'_{gH} is ${}^{CF}[(i)-gH]$ -differentiable. From situation (K1) of Lemma 3.1, boundary value problems (1.1)–(1.2) are equivalent to integral equation

$$u(t) = \Lambda_1 \oplus u_0 t \oplus \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds, \quad t \in [0, T].$$

Then, we define the operator $A_1 : B \rightarrow S$ by

$$(A_1 u)(t) := \Lambda_1 \oplus u_0 t \oplus \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds, \quad t \in [0, T].$$

In order to apply Lemma 2.11, we separate the proof into the following three steps.

Step 1. A_1 maps B into itself. For any $u \in B$ and $t \in [0, T]$, from Lemmas 2.3 and 2.6 as well as (H2), we have

$$\begin{aligned} d(A_1u(t), \widehat{0}) &= d\left(A_1 \oplus u_0t \oplus \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds, \widehat{0}\right) \\ &\leq \frac{\lambda}{1-\lambda} d(u_0T, \widehat{0}) + \frac{\lambda}{(1-\lambda)\Gamma(q)} d\left(\int_0^T (T-s)^{q-1} f(s, u(s)) ds, \widehat{0}\right) \\ &\quad + d(u_0t, \widehat{0}) + \frac{1}{\Gamma(q)} d\left(\int_0^t (t-s)^{q-1} f(s, u(s)) ds, \widehat{0}\right) \\ &\leq \frac{\lambda}{1-\lambda} d(u_0T, \widehat{0}) + \frac{\lambda}{(1-\lambda)\Gamma(q)} \int_0^T (T-s)^{q-1} d(f(s, u(s)), \widehat{0}) ds \\ &\quad + d(u_0t, \widehat{0}) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} d(f(s, u(s)), \widehat{0}) ds \\ &\leq \frac{\lambda}{1-\lambda} d(u_0T, \widehat{0}) + \frac{\lambda M}{(1-\lambda)\Gamma(q)} \int_0^T (T-s)^{q-1} ds + d(u_0t, \widehat{0}) + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds \\ &= \frac{\lambda}{1-\lambda} d(u_0T, \widehat{0}) + \frac{\lambda M}{(1-\lambda)\Gamma(q+1)} T^q + d(u_0t, \widehat{0}) + \frac{M}{\Gamma(q+1)} t^q \\ &\leq \frac{T}{1-\lambda} d(u_0, \widehat{0}) + \frac{MT^q}{(1-\lambda)\Gamma(q+1)} \leq R. \end{aligned}$$

It follows that $D(A_1u, \widehat{0}) \leq R$. Thus, A_1 maps B into itself.

Step 2. A_1 is a continuous operator on B . For this, let $u_n \rightarrow u$ ($n \rightarrow \infty$) in B . From Lemmas 2.3 and 2.6, we have

$$\begin{aligned} &d(A_1u_n(t), A_1u(t)) \\ &\leq \frac{\lambda}{(1-\lambda)\Gamma(q)} d\left(\int_0^T (T-s)^{q-1} f(s, u_n(s)) ds, \int_0^T (T-s)^{q-1} f(s, u(s)) ds\right) \\ &\quad + \frac{1}{\Gamma(q)} d\left(\int_0^t (t-s)^{q-1} f(s, u_n(s)) ds, \int_0^t (t-s)^{q-1} f(s, u(s)) ds\right) \\ &\leq \frac{\lambda}{(1-\lambda)\Gamma(q)} \int_0^T (T-s)^{q-1} d(f(s, u_n(s)), f(s, u(s))) ds \\ &\quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} d(f(s, u_n(s)), f(s, u(s))) ds \\ &\leq \frac{\lambda T^q}{(1-\lambda)\Gamma(q+1)} \sup_{t \in [0, T]} d(f(t, u_n(t)), f(t, u(t))) + \frac{t^q}{\Gamma(q+1)} \sup_{t \in [0, T]} d(f(t, u_n(t)), f(t, u(t))) \\ &\leq \frac{T^q}{(1-\lambda)\Gamma(q+1)} \sup_{t \in [0, T]} d(f(t, u_n(t)), f(t, u(t))). \end{aligned}$$

According to (H1), f is continuous on $[0, T] \times \mathbb{E}_c^1$. So, $\sup_{t \in [0, T]} d(f(t, u_n(t)), f(t, u(t))) \rightarrow 0$ when $n \rightarrow \infty$.

We get $D(A_1u_n, A_1u) \rightarrow 0$ when $n \rightarrow \infty$. So A_1 is a continuous operator on B .

Step 3. $A_1(B)$ is relatively compact in $C^\mathbb{E}([0, T], \mathbb{E}_c^1)$. According to Lemma 2.4, we just need to prove

- (i) $A_1(B)$ is equicontinuous;
- (ii) for every $t \in [0, T]$, the set $A_1(B)(t)$ is relatively compact in \mathbb{E}_c^1 .

Firstly, we verify the equicontinuity of $A_1(B)$. Let $t_1, t_2 \in [0, T]$, $t_1 < t_2$ and $u \in B$. From Lemmas 2.3 and 2.6 as well as (H2), we get

$$\begin{aligned} & d(A_1u(t_2), A_1u(t_1)) \\ &= d\left(u_0t_2 \oplus \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2 - s)^{q-1} f(s, u(s)) ds, u_0t_1 \oplus \frac{1}{\Gamma(q)} \int_0^{t_1} (t_2 - s)^{q-1} f(s, u(s)) ds\right) \\ &\leq d\left(\frac{1}{\Gamma(q)} \int_0^{t_2} (t_2 - s)^{q-1} f(s, u(s)) ds, \frac{1}{\Gamma(q)} \int_0^{t_1} (t_2 - s)^{q-1} f(s, u(s)) ds\right) + d(u_0t_2, u_0t_1) \\ &= \frac{1}{\Gamma(q)} d\left(\int_0^{t_1} (t_2 - s)^{q-1} f(s, u_n(s)) ds \oplus \int_{t_1}^{t_2} (t_2 - s)^{q-1} f(s, u_n(s)) ds, \int_0^{t_1} (t_1 - s)^{q-1} f(s, u(s)) ds\right) \\ &\quad + d(u_0t_2, u_0t_1) \\ &\leq \frac{1}{\Gamma(q)} \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] d(f(s, u(s)), \widehat{0}) ds + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} d(f(s, u(s)), \widehat{0}) ds \\ &\quad + d(u_0t_2, u_0t_1) \\ &\leq \frac{M}{\Gamma(q)} \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] ds + \frac{M}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} ds + (t_2 - t_1) d(u_0, \widehat{0}) \\ &= \frac{M}{\Gamma(q+1)} [2(t_2 - t_1)^q + (t_2^q - t_1^q)] + (t_2 - t_1) d(u_0, \widehat{0}). \end{aligned}$$

This implies that $d(A_1u(t_2), A_1u(t_1)) \rightarrow 0$ when $|t_2 - t_1| \rightarrow 0$. Hence $A_1(B)$ is an equicontinuous subset of $C^{\mathbb{E}}([0, T], \mathbb{E}_c^1)$.

Secondly, we prove that $A_1(B)(t)$ is relatively compact in \mathbb{E}_c^1 . On the one hand, due to the relative compactness of $f([0, T] \times B)$, there exists compact set \widetilde{K} such that $[f(t, u)]^0 \subseteq \widetilde{K}$ for all $(t, u) \in [0, T] \times B$. So

$$\begin{aligned} [A_1(B)(t)]^0 &= \left[A_1 \oplus u_0t \oplus \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s, u(s)) ds\right]^0 \\ &= \frac{\lambda T}{1-\lambda} [u_0]^0 + \frac{\lambda}{(1-\lambda)\Gamma(q)} \int_0^T (T - s)^{q-1} [f(s, u(s))]^0 ds \\ &\quad + t[u_0]^0 + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} [f(s, u(s))]^0 ds \\ &\subseteq \left[\frac{\lambda}{(1-\lambda)\Gamma(q)} \int_0^T (T - s)^{q-1} ds + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} ds\right] \widetilde{K} + \frac{T}{1-\lambda} [u_0]^0 \\ &= \left[\frac{\lambda T^q}{(1-\lambda)\Gamma(q+1)} + \frac{t^q}{\Gamma(q+1)}\right] \widetilde{K} + \frac{T}{1-\lambda} [u_0]^0. \end{aligned}$$

Therefore, there exists a compact set $K_0 \subseteq \mathbb{R}$ such that $[A_1(B)(t)]^0 \subseteq K_0$, that is, $A_1(B)(t)$ is compact-supported for every $t \in [0, T]$. On the other hand, fixing $t \in [0, T]$, we have $A_1(B)(t) \in \mathbb{E}_c^1$. If $y \in A_1(B)(t)$, then $y = A_1(u)(t)$ for some $u \in B$. So, we have

$$[y]^\alpha = A_1 \oplus t[u_0]^\alpha \oplus \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} [f(s, u(s))]^\alpha ds, \quad \alpha \in [0, 1].$$

By Lemma 2.5, we know that $f([0, T] \times B)$ is level-equicontinuous. Hence for any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$d_H([f(t, u)]^\alpha, [f(t, u)]^\beta) \leq \frac{(1 - \lambda)\Gamma(q + 1)\varepsilon}{2T^q} \text{ for all } (t, u) \in [0, T] \times B$$

and $d_H([u_0]^\alpha, [u_0]^\beta) \leq \frac{(1-\lambda)\varepsilon}{2T}$ when $|\alpha - \beta| < \delta$. So for all $|\alpha - \beta| < \delta$ and $(t, u) \in [0, T] \times B$, one gets

$$\begin{aligned} & d_H([y]^\alpha, [y]^\beta) \\ &= d_H([A_1 u(t)]^\alpha, [A_1 u(t)]^\beta) \\ &\leq \frac{\lambda T}{1-\lambda} d_H([u_0]^\alpha, [u_0]^\beta) + \frac{\lambda}{(1-\lambda)\Gamma(q)} \int_0^T (T-s)^{q-1} d_H\left([f(s, u(s))]^\alpha, [f(s, u(s))]^\beta\right) ds \\ &\quad + t d_H([u_0]^\alpha, [u_0]^\beta) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} d_H\left([f(s, u(s))]^\alpha, [f(s, u(s))]^\beta\right) ds \\ &\leq \frac{\lambda T}{1-\lambda} d_H([u_0]^\alpha, [u_0]^\beta) + \frac{\lambda T^q}{(1-\lambda)\Gamma(q+1)} \cdot \frac{(1-\lambda)\Gamma(q+1)\varepsilon}{2T^q} \\ &\quad + t d_H([u_0]^\alpha, [u_0]^\beta) + \frac{t^q}{\Gamma(q+1)} \cdot \frac{(1-\lambda)\Gamma(q+1)\varepsilon}{2T^q} \leq \varepsilon, \end{aligned}$$

which means $A_1(B)(t)$ is level-equicontinuous in \mathbb{E}_c^1 on $[0, 1]$ for each $t \in [0, T]$. Therefore, $A_1(B)(t)$ is a relatively compact subset of \mathbb{E}_c^1 from Lemma 2.5. Then according to Lemma 2.4, we obtain that $A_1(B)$ is relatively compact in $C^\mathbb{E}([0, T], \mathbb{E}_c^1)$.

Through the above elaboration, we conclude that A_1 is a compact operator. By Lemma 2.11, it follows that A_1 has at least one fixed point in B . Equivalently, there is at least one ${}^{CF}[(i-i)-gH]$ -differentiable solution to boundary value problems (1.1)–(1.2).

Case 2. u is (i)- gH -differentiable and u'_{gH} is ${}^{CF}[(ii)-gH]$ -differentiable. From situation (K2) of Lemma 3.1, boundary value problems (1.1)–(1.2) are equivalent to integral equation

$$u(t) = \Lambda_2 \oplus u_0 t \ominus \frac{-1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds, \quad t \in [0, T].$$

Then, we define the operator $A_2 : B \rightarrow S$ by

$$(A_2 u)(t) = \Lambda_2 \oplus u_0 t \ominus \frac{-1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds, \quad t \in [0, T].$$

For any $u \in B$ and $t \in [0, T]$, from Lemmas 2.3 and 2.6 as well as (H2), we have

$$\begin{aligned} d(A_2 u(t), \widehat{0}) &= d\left(\Lambda_2 \oplus u_0 t \ominus \frac{-1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds, \widehat{0}\right) \\ &\leq \frac{\lambda}{1-\lambda} d(u_0 T, \widehat{0}) + \frac{\lambda}{(1-\lambda)\Gamma(q)} d\left(\int_0^T (T-s)^{q-1} f(s, u(s)) ds, \widehat{0}\right) \\ &\quad + d(u_0 t, \widehat{0}) + \frac{1}{\Gamma(q)} d\left(\int_0^t (t-s)^{q-1} f(s, u(s)) ds, \widehat{0}\right) \\ &\leq \frac{T}{1-\lambda} d(u_0, \widehat{0}) + \frac{MT^q}{(1-\lambda)\Gamma(q+1)} \leq R, \end{aligned}$$

which means A_2 maps B into itself. The remainder of the argument is analogous to that in Case 1 but will not be reproduced here. The proof is completed. \square

Theorem 3.2. *Suppose $\lambda \in (0, 1)$ and f satisfies (H1)–(H2) as well as the following condition:*

(H3) *there exists a constant $L \geq 0$, for all $u, v \in \mathbb{E}_c^1$, such that*

$$d(f(t, u), f(t, v)) \leq Ld(u, v).$$

Then boundary value problems (1.1)–(1.2) have a unique ${}^{CF}[(i-i)-gH]$ -differentiable solution or ${}^{CF}[(i-ii)-gH]$ -differentiable solution on $[0, T]$.

Proof. Since $\lambda \in (0, 1)$ and f satisfies (H1)–(H2), it follows that boundary value problems (1.1)–(1.2) have at least one ${}^{CF}[(i-i)-gH]$ -differentiable solution or ${}^{CF}[(i-ii)-gH]$ -differentiable solution according to Theorem 3.1. In the sequel, we only prove ${}^{CF}[(i-i)-gH]$ -differentiable case. The proof of the other case is similar.

Suppose there exist two ${}^{CF}[(i-i)-gH]$ -differentiable solutions u and v to boundary value problems (1.1)–(1.2). Then, for all $t \in [0, T]$, we have

$$\begin{aligned} d(u(t), v(t)) &= d(A_1 u(t), (A_1 v(t))) \\ &\leq \frac{\lambda}{(1-\lambda)\Gamma(q)} d\left(\int_0^T (T-s)^{q-1} f(s, u(s)) ds, \int_0^T (T-s)^{q-1} f(s, v(s)) ds\right) \\ &\quad + \frac{1}{\Gamma(q)} d\left(\int_0^t (t-s)^{q-1} f(s, u(s)) ds, \int_0^t (t-s)^{q-1} f(s, v(s)) ds\right) \\ &\leq \frac{\lambda}{(1-\lambda)\Gamma(q)} \int_0^T (T-s)^{q-1} d(f(s, u(s)), f(s, v(s))) ds \\ &\quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} d(f(s, u(s)), f(s, v(s))) ds \\ &\leq \left[\frac{\lambda L}{(1-\lambda)\Gamma(q)} + \frac{L}{\Gamma(q)} \right] \int_0^T (T-s)^{q-1} d(u(s), v(s)) ds. \end{aligned}$$

Therefore, we obtain $u(t) = v(t)$ for all $t \in [0, T]$ according to Lemma 2.12. Hence the solution is unique. The proof is completed. \square

Corollary 3.2. Suppose $\lambda = 0$. For the initial value problem (3.7), we also have corresponding results if it satisfies the conditions of Theorem 3.1 or 3.2.

Theorem 3.3. Suppose that $\lambda \in (1, +\infty)$ and right-hand side function f satisfies (H1)–(H2). Then boundary value problems (1.1)–(1.2) have at least one ${}^{CF}[(ii-i)-gH]$ -differentiable solution or ${}^{CF}[(ii-ii)-gH]$ -differentiable solution.

Proof. For the set B defined as (3.8), choose $R = \frac{(2\lambda-1)T}{\lambda-1} d(u_0, \widehat{0}) + \frac{(2\lambda-1)MT^q}{(\lambda-1)\Gamma(q+1)} + 1$. If u is (ii)- gH -differentiable and u'_{gH} is ${}^{CF}[(i)-gH]$ -differentiable, then we consider the operator $A_3 : B \rightarrow S$ defined by

$$(A_3 u)(t) = A_1 \ominus (-1) \left[u_0 t \oplus \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds \right].$$

If u is (ii)- gH -differentiable and u'_{gH} is ${}^{CF}[(ii)-gH]$ -differentiable, then can we consider the operator $A_4 : B \rightarrow S$ defined by

$$(A_4 u)(t) = A_2 \ominus (-1) \left[u_0 t \ominus \frac{-1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds \right].$$

The remainder of the argument is analogous to the proof of Theorem 3.1. The proof is completed. \square

Similar to Theorem 3.2, we can get the following uniqueness results.

Theorem 3.4. Suppose $\lambda \in (1, +\infty)$ and f satisfies (H1)–(H3). Then boundary value problems (1.1)–(1.2) have a unique ${}^{CF}[(ii-i)-gH]$ -differentiable solution or ${}^{CF}[(ii-ii)-gH]$ -differentiable solution on $[0, T]$.

4 Example

In this section, we will present an example to illustrate our main results. Consider the following boundary value problem

$$\begin{cases} {}^C_{gH} \mathfrak{D}_*^{\frac{3}{2}} u(t) = \sin u(t) \oplus \eta e^{-t}, & 0 < t < 1, \\ u(0) = 0.75u(1), \quad u'_{gH}(0) = \widehat{0}, \end{cases} \tag{4.1}$$

where fuzzy sinusoidal function $\sin u(t)$ is obtained by Zadeh's extension principle, and fuzzy constants $\eta = (0.5, 1, 1.5)$ and $\widehat{0} = (0, 0, 0)$ are two symmetric triangular fuzzy numbers. Let $f(t, u) = \sin u + \eta e^{-t}$. Apparently, f is a continuous and compact fuzzy-valued function. For $t \in [0, 1]$, from the definition of $d(u, v)$ and Lemma 2.2, we can easily get

$$\begin{aligned} M &= \sup_{(t,u) \in [0,1] \times \mathbb{E}_c^1} d(f(t, u), \widehat{0}) \\ &= \sup_{(t,u) \in [0,1] \times \mathbb{E}_c^1} d(\sin u(t) \oplus \eta e^{-t}, \widehat{0}) \\ &\leq \sup_{(t,u) \in [0,1] \times \mathbb{E}_c^1} d(\sin u(t), \widehat{0}) + \sup_{(t,u) \in [0,1] \times \mathbb{E}_c^1} d(\eta e^{-t}, \widehat{0}) \\ &\leq \sup_{(t,u) \in [0,1] \times \mathbb{E}_c^1} d(\sin u(t), \widehat{0}) + \sup_{(t,u) \in [0,1] \times \mathbb{E}_c^1} d(\eta, \widehat{0}) \\ &= 1 + 1.5 = 2.5. \end{aligned}$$

So, we take $R = \frac{2.5}{0.25\Gamma(2.5)} + 1 \approx 8.5225$. By Theorem 3.1, the boundary value problem (4.1) has at least one ${}^{CF}[(i-i)-gH]$ -differentiable solution. At the same time, the Hukuhara difference $u_0 t \ominus \frac{-1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds$ does not exist for any $t \in (0, 1]$. It means that problem (4.1) has no ${}^{CF}[(i-ii)-gH]$ -differentiable solution. In addition, due to

$$\begin{aligned} d(f(t, u), f(t, v)) &= d(\sin u \oplus \eta e^{-t}, \sin v \oplus \eta e^{-t}) \\ &= d(\sin u, \sin v) \\ &\leq d(u, v) \end{aligned}$$

for all $u, v \in \mathbb{E}^1$, we know the ${}^{CF}[(i-i)-gH]$ -differentiable solution is unique by Theorem 3.2.

5 Example

The most significant part of this study is our research object. This work regarding FFDEs with order $q \in (1, 2]$ is novel. Another important part is that some other kinds of fuzzy boundary value problems can also be made under the approach similarly to the above mentioned procedure. The authors of this article strongly believe that this article will also motivate some physicists, engineers and scholars to make further researches in this area. In this respect, there may be other extended work.

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