Some Embedding Theorems on the Nikolskii-Morrey Type Spaces

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Abstract In the paper the Nikolskii-Morrey type spaces $H^{l}_{p,\varphi,\beta}(G)$ were introduced and studied. Some embedding theorems are obtained in $H^{l}_{p,\varphi,\beta}(G)$ with the help of Nikolskii type integral representation.

Keywords: Nikolskii-Morrey type spaces, integral representation, embedding theorems, generalized Holder condition

1 Introduction

In the paper, we introduce a Nikolskii-Morrey type space with parameters. By $H^{l}_{p,\varphi,\beta}(G)$ we denote the spaces of all functions $f \in L^{1}_{loc}(G)$ ($m_{i} > l_{i} - k_{i} > 0, i = 1, 2, ..., n$) with the finite norm

$$\|f\|_{H^{l}_{p,\varphi,\beta}(G)} = \|f\|_{L^{p,\varphi,\beta}(G)} + \sum_{i=1}^{n} \sup_{0 < h < h_{0}} \left| \Delta^{m_{i}}_{l_{i}} (\varphi_{l_{i}}(h), G_{\varphi_{l_{i}}(h)}) D_{l_{i}}^{k_{i}} f \right|_{p,\varphi,\beta},$$

(1.1)

where

$$\|f\|_{p,\varphi,\beta;G} = \|f\|_{L^{p,\varphi,\beta}(G)} = \sup_{x \in G \cap t > 0} \left( |\varphi((t_{1})|^{-\beta} \|f\|_{p,\varphi,\beta}(x)) \right),$$

(1.2)

$l \in (0, \infty)^{n}$, $m_{i} \in \mathbb{N}$, $k_{i} \in \mathbb{N}_{0}$, $p \in [1, \infty)$, $[t_{1}] = \min \{1, t\}$, $\varphi(t) = (\varphi_{1}(t), ..., \varphi_{n}(t))$, $|\varphi((t_{1})|^{-\beta} = \prod_{j=1}^{n} (\varphi_{j}((t_{1})|^{-\beta}$, $\beta_{j} \in [0, 1], j = 1, 2, ..., n$.

Denote by $\varphi$ the set of vector functions $\varphi(t) = (\varphi_{1}(t), ..., \varphi_{n}(t))$ with Lebesgue measurable functions $\varphi_{j}(t) > 0, t > 0, \lim_{t \to t_{0}^{+}} \varphi_{j}(t) = 0, \lim_{t \to \infty} \varphi_{j}(t) = \infty, j = 1, 2, ..., n$.

For any $x \in \mathbb{R}^{n}$,

$$G_{\varphi(t)}(x) = G \cap I_{\varphi}(t) = G \cap \left\{ y : |y_{j} - x_{j}| < \frac{1}{2} \varphi_{j}(t), \quad j = 1, 2, ..., n \right\}.$$

Let for any $t > 0, |\varphi((t_{1})| \leq C$, where $C$ is positive constant. Then the embeddings $L^{p,\varphi,\beta}(G) \to L^{p}(G)$ and $H^{l}_{p,\varphi,\beta}(G) \to H^{l}_{p}(G)$ hold, i.e.

$$\|f\|_{p,\varphi;G} \leq c \|f\|_{p,\varphi,\beta;G},$$

$$\|f\|_{H^{l}_{p}(G)} \leq c \|f\|_{H^{l}_{p,\varphi,\beta}(G)}.$$

Note that the spaces $L^{p,\varphi,\beta}(G)$ and $H^{l}_{p,\varphi,\beta}(G)$ are Banach spaces. The completeness of these spaces automatically implies from completeness of $L^{p}$ and $H^{l}_{p}$. The space $H^{l}_{p,\varphi,\beta}(G)$, when $\varphi_{j}(t) = t^{\alpha_{j}}$, $\beta_{j} = \alpha_{j} p$, ($j = 1, ..., n$) coincides with the space $H^{l}_{p,\alpha,\chi}(G) \equiv H^{l}_{p,\alpha,\chi}(G)$ introduced by J.Ross [13], in the case $\beta_{j} = \alpha_{j} p$, ($j = 1, ..., n$) it coincides with the Nikolski space $H^{l}_{p}(G)$. The space $W^{l}_{p,\varphi,\beta}(G)$ was introduced and studied in [12]. The spaces of such type with different norms were introduced and studied in [2]–[11].

Note some properties of the spaces $L^{p,\varphi,\beta}(G)$.

1. The space $L^{p,\varphi,\beta}(G)$ is complete.

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Proof. Let \( \{f_i\}_{i=1}^{\infty} \) be the fundamental sequences in \( L_{p,\varphi,\beta}(G) \), i.e., for any \( i, j \to \infty \)

\[
\|f_i - f_j\|_{L_{p,\varphi,\beta}(G)} \to 0,
\]

It means that \( \forall \varepsilon > 0, \exists n_0 \in N, \forall i, j > n_0 \)

\[
\|f_i - f_j\|_{L_{p,\varphi,\beta}(G)} < \varepsilon,
\]

In other words \( \forall \varepsilon > 0, \exists n_0 \in N, \forall i, j > n_0 \)

\[
\sup_{x \in G, t > 0} \left( |\varphi([t]_1)|^{-\beta} \|f_i - f_j\|_{L_{p,G_{\varphi(t)}(x)}} \right) < \varepsilon
\]

and for any \( x \in G, \forall t > 0 \)

\[
\|f_i - f_j\|_{L_{p,G_{\varphi(t)}(x)}} < \varepsilon
\]

i.e. \( \{f_i\}_{i=1}^{\infty} \) is a Cauchy sequence in \( L_p(G_{\varphi(t)}(x)) \). The space \( L_p(G) \) is complete, therefore there is a function \( f_0 \in L_p(G) \), \( i \to \infty \), for \( x \in G \), for any \( t > 0, \forall \varepsilon > 0 \)

\[
\|f_i - f_0\|_{L_{p,G_{\varphi(t)}(x)}} \to \varepsilon
\]

then

\[
|\varphi([t]_1)|^{-\beta} \|f_i - f_0\|_{L_{p,G_{\varphi(t)}(x)}} \to \varepsilon
\]

i.e.

\[
\|f_i - f_0\|_{p,\varphi,\beta,G} < \varepsilon,
\]

\[
\|f_0\|_{p,\varphi,\beta,G} = \|f_i - f_0\|_{L_{p,\varphi,\beta}(G)} + \|f_i\|_{L_{p,\varphi,\beta}(G)} < \varepsilon_1 + M = \varepsilon_2
\]

\[
\|f_0\|_{p,\varphi,\beta,G} < \varepsilon_2, f_0 \in L_{p,\varphi,\beta}(G).
\]

2. Let \( G \) be a bounded domain and \( p \leq q; \varphi(t) \leq \psi(t)(t > 0) \); \( \exists c > 0, \forall t \in (0,1), |\psi(t)|^\beta \leq c|\varphi(t)|^\beta \), and then \( L_{q,\psi,\beta}(G) \to L_{p,\varphi,\beta}(G) \) and there exists \( C > 0 \) such that

\[
\|f\|_{p,\varphi,\beta,G} \leq C\|f\|_{q,\psi,\beta,G}.
\]

Proof. For any \( t > 0, x \in G \) we have

\[
|\varphi([t]_1)|^{-\beta} \|f\|_{p,G_{\varphi(t)}(x)}
\]

\[
\leq |\varphi([t]_1)|^{-\beta} (mesG_{\varphi(t)}(x))^{\frac{1}{\beta}} \frac{1}{|\psi([t]_1)|^{\beta}} |\psi([t]_1)|^{-\beta} \|f\|_{q,G_{\varphi(t)}(x)}
\]

and

\[
\|f\|_{p,\varphi,\beta,G} \leq C\|f\|_{q,\psi,\beta,G}.
\]

Definition 1.1. The open set \( G \subset \mathbb{R}^n \) is said to be an open set with condition of flexible \( \varphi \)-horn if for some \( \theta \in (0,1]^n, T \in (0,\infty) \) for any \( x \in G \) there exists the vector-function

\[
\rho (\varphi(t), x) = (\rho_1 (\varphi_1(t), x), \ldots, \rho_n (\varphi_n(t), x)), \ 0 \leq t \leq T
\]

with the following properties:

1) For all \( j = 1, 2, \ldots, n \), \( \rho_j (\varphi_j(t), x) \) are absolutely continuous on \([0,T] \), \( |\rho_j (\varphi_j(t), x)| \leq 1 \) for almost all \( t \in [0,T], \)

2) \( \rho_j (0, x) = 0, \ x + V(x, \theta) = x + \bigcup_{0 \leq t \leq T} [\rho (\varphi(t), x) + \varphi(t) \theta I] \subset G. \)

In particular, for \( \varphi(t) = t^\lambda, (t^\lambda = (t^{\lambda_1}, t^{\lambda_2}, \ldots, t^{\lambda_n})) \) and \( \theta_j = \theta^{\lambda_j} \) \( (j = 1, \ldots, n) \) the set \( x + V(x, \theta) \) is called the flexible \( \lambda \)-horn introduced in [1].
Assuming that $\varphi_j(t)$ ($j = 1, 2, ..., n$) are also differentiable on $[0, T]$, we can show that for $f \in H^1_0(G)$ determined in $n$-dimensional domains, satisfying the condition of flexible $\varphi$-horn, it holds the following integral representation ($\forall x \in U \subset G$)

\[
D^\nu f(x) = J_{\varphi(t)}^{(\nu)}(x) + (-1)^{[\nu]} \sum_{i=1}^{n} \int_{0}^{T} \int_{R^n} K_i^{(\nu)} \left( \frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)} \right) \times \zeta_i \left( \frac{u_i(\varphi_i(t), x)}{\varphi_i(t)} - \frac{1}{2} \rho_i'(\varphi_i(t), x) \right) \Delta_{\eta_i}(\varphi_i(\delta) u) \\
\times f(x + y + u e_i) \prod_{j=1}^{n} (\varphi_j(t))^{-\nu_j - 2} \frac{\varphi_j'(t)}{\varphi_j(t)} dt du dy,
\]

(1.3)

\[
J_{\varphi(t)}^{(\nu)}(x) = \prod_{j=1}^{n} \varphi_j^{-2\nu_j}(t) \int_{R^n} \int_{R^n} \Theta^{(\nu)} \left( \frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{2\varphi(t)} \right) \times \Omega \left( \frac{z}{\varphi(T)}, \frac{\rho(\varphi(t), x)}{2\varphi(t)} \right) f(x + y + z) dy dz.
\]

(1.4)

Let $M_i (., y) \in C_0^\infty (R^n)$ be such that

\[
S(M_i) \subset I_{\varphi(t)} = \left\{ y : |y_j| < \frac{1}{2} \varphi_j(t), \; j = 1, 2, ..., n \right\}.
\]

Assume $0 < T \leq 1$ is fixed and

\[
V = \bigcup_{0 < t \leq T} \left\{ y : \frac{y}{\varphi(t)} \in S(M_i) \right\}.
\]

It is clear that $V \subset I_{\varphi(t)}$. Let $U + V \subset G$.

**Lemma 1.2.** Let $1 \leq p \leq q \leq r \leq \infty$; $0 < \eta$, $t < T \leq 1$, $\nu = (\nu_1, \nu_2, ..., \nu_n)$, $\nu_j \geq 0$ are integers, $j = 1, 2, ..., n$; $\Delta_{\eta_i}^{(\nu_i)}(h) \in L_{p, \varphi, \beta}(G)$ and let

\[
Q_T^\nu = \int_{0}^{T} \prod_{j=1}^{n} (\varphi_j(t))^{-\nu_j - (1-\beta, p)(\frac{1}{\beta} - \frac{1}{2} \varphi_j(t))^{-\nu_j - 1} - \frac{\varphi_j'(t)}{\varphi_j(t)} dt < \infty,
\]

\[
A(x) = \prod_{j=1}^{n} \int_{R^n} \int_{R^n} f(x + y + z) \Omega^{\nu} \left( \frac{y}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{2\varphi(t)} \right) \times \Omega \left( \frac{z}{\varphi(T)}, \frac{\rho(\varphi(t), x)}{2\varphi(t)} \right) f(x + y + z) dy dz,
\]

(1.5)

\[
H^{\eta}_n(x) = \int_{0}^{\eta} \int_{R^n} \prod_{j=1}^{n} (\varphi_j(t))^{\nu_j - 2} \frac{\varphi_j'(t)}{\varphi_j(t)} dt,
\]

(1.6)

\[
H^{\eta}_{\nu T}(x) = \int_{\eta}^{T} \int_{R^n} \prod_{j=1}^{n} (\varphi_j(t))^{\nu_j - 2} \frac{\varphi_j'(t)}{\varphi_j(t)} dt,
\]

(1.7)

where

\[
L_i(x, t) = \int_{R^n} \int_{-\infty}^{+\infty} M_i \left( \frac{y}{\varphi_i(t)}, \frac{\rho(\varphi_i(t), x)}{\varphi_i(t)} \right) \times \zeta_i \left( \frac{u_i(\varphi_i(t), x)}{\varphi_i(t)} - \frac{1}{2} \rho_i'(\varphi_i(t), x) \right) \Delta_{\eta_i}^{(\nu_i)}(\varphi_i(\delta) u) f(x + y + u e_i) dy du
\]

(1.8)
Then for any \( \varphi \in U \) the following inequalities

\[
\sup_{\varphi \in U} \| H_{\varphi} f \|_{q_{U_{\psi}(\xi)}} \leq C_1 \left\| \left( \varphi_i(t) \right)^{-r_i} \Delta_{i}^{n_i} \left( \varphi_i(t), G_{\varphi_i(t)} \right) f \right\|_{p, \varphi, \beta, G}
\]

\[
\times |Q_{\varphi}^{j} \prod_{j=1}^{n} (\psi_j (\{ \xi \}_1))^{\beta_j} |
\]

\[ (1.9) \]

\[
\sup_{\varphi \in U} \| H_{\varphi}^{T} f \|_{q_{U_{\psi}(\xi)}} \leq C_2 \left\| \left( \varphi_i(t) \right)^{-r_i} \Delta_{i}^{n_i} \left( \varphi_i(t), G_{\varphi_i(t)} \right) f \right\|_{p, \varphi, \beta, G}
\]

\[
\times |Q_{\varphi}^{j} \prod_{j=1}^{n} (\psi_j (\{ \xi \}_1))^{\beta_j} |
\]

\[ (1.10) \]

\[
\sup_{\varphi \in U} \| A \|_{q_{U_{\psi}(\xi)}} \leq \| f \|_{p, \varphi, \beta, G} \prod_{j=1}^{n} (\varphi_j (t))^{-\nu_j - (1-\beta_j) \rho_j (\frac{1}{2} - \frac{1}{p})} \prod_{j=1}^{n} (\psi_j (\{ \xi \}_1))^{\beta_j}
\]

\[ (1.11) \]

is hold, where \( U_{\varphi}(\xi) = \{ x : |x - \varphi| < \frac{1}{2} \psi_j (\xi), j = 1, 2, ..., n \} \) and \( \psi \in N, \ C_1, C_2 \) are the constants independent of \( \varphi, \xi, \eta \) and \( T \).

**Proof.** Applying sequentially the Minkowski generalized inequality for any \( \varphi \in U \)

\[
\left\| H_{\varphi} f \right\|_{q_{U_{\psi}(\xi)}} \leq \int_{0}^{\eta} \left\| L_i (\cdot, t) \right\|_{q_{U_{\psi}(\xi)}} \prod_{j=1}^{n} (\varphi_j (t))^{-\nu_j - 2 \varphi_j (t)} \frac{\varphi_j (t)}{\varphi_i (t)} dt,
\]

and from the Hölder inequality \((q \leq r)\) we have

\[
\left\| L_i (\cdot, t) \right\|_{q_{U_{\psi}(\xi)}} \leq \left\| L_i (\cdot, t) \right\|_{r_{U_{\psi}(\xi)}} \prod_{j=1}^{n} (\psi_j (\xi))^{\frac{1}{q} - \frac{1}{r}}.
\]

(1.13)

Now estimate the norm \( \left\| L_i (\cdot, t) \right\|_{q_{U_{\psi}(\xi)}} \). Let \( X \) be a characteristic function of the set \( S (M_i) = \text{sup} \ M_i \). Noting that \( 1 \leq p \leq r \leq \infty, s \leq r \), represent the integrand function (1.8) in the form

\[
\int_{-\infty}^{+\infty} M_i \zeta_{\varphi} \Delta_{i}^{n_i} f du = \left( \int_{-\infty}^{+\infty} \zeta_{\varphi} \Delta_{i}^{n_i} f du \right)^{\frac{s}{p}} \left( \int_{-\infty}^{+\infty} \zeta_{\varphi} \Delta_{i}^{n_i} f du \right)^{\frac{1}{q} - \frac{1}{p}} \right)
\]

\[
\times \left( \int_{-\infty}^{+\infty} \zeta_{\varphi} \Delta_{i}^{n_i} f du \right)^{\frac{1}{p}} X \left( \frac{y}{\varphi_i(t)} \right) \left( \frac{y}{\varphi_i(t)} \right)^{\frac{s}{p} - \frac{1}{q}}
\]

and apply to \( |L_i| \) the Hölder inequality \( \left( \frac{1}{p} + \left( \frac{1}{p} - \frac{1}{q} \right) + \left( \frac{1}{q} - \frac{1}{p} \right) = 1 \) \), we obtain

\[
\left\| L_i (\cdot, t) \right\|_{q_{U_{\psi}(\xi)}} \leq \sup_{\varphi \in U_{\psi}(\xi)} \left( \int_{R^n} \int_{-\infty}^{+\infty} \zeta_{\varphi} \left( \frac{u}{\varphi_i(t)} \right) \rho_i \left( \varphi_i(t), x \right) \frac{1}{2} \rho_i \left( \varphi_i(t), x \right) \right)
\]

\[
\times \Delta_{i}^{n_i} (\varphi_i(t)) f (x + y + u e_i) du \right)^{\frac{1}{p} - \frac{1}{p}} \right)
\]

\[
\times \sup_{\varphi \in U_{\psi}(\xi)} \left( \int_{R^n} \int_{-\infty}^{+\infty} \zeta_{\varphi} \left( \frac{u}{\varphi_i(t)} \right) \rho_i \left( \varphi_i(t), x \right) \frac{1}{2} \rho_i \left( \varphi_i(t), x \right) \right)
\]

\[
\times \Delta_{i}^{n_i} (\varphi_i(t)) f (x + y + u e_i) du \right)^{\frac{1}{p} - \frac{1}{p}} \right)
\]

\[
\times \left( \int_{-\infty}^{+\infty} \zeta_{\varphi} \Delta_{i}^{n_i} f du \right)^{\frac{1}{p}} X \left( \frac{y}{\varphi_i(t)} \right) \left( \frac{y}{\varphi_i(t)} \right)^{\frac{s}{p} - \frac{1}{q}}
\]
For any $x \in U$ we have
\[
\int_{\mathbb{R}^n} \left| \int_{-\infty}^{+\infty} \zeta_i \left( \frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t), x)}{\varphi_i(t)} - \frac{1}{2} \rho_i'(\varphi_i(t), x) \right) \Delta^m_{i} (\varphi_i(\delta) u) f (x + y + u \epsilon_i) dy \right| \, dx
\]
\[
\leq \int_{(U + V) \cup (\mathbb{C} \setminus \mathbb{R})} \left| \int_{-\infty}^{+\infty} \zeta_i \left( \frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t), x)}{\varphi_i(t)} - \frac{1}{2} \rho_i'(\varphi_i(t), x) \right) \Delta^m_{i} (\varphi_i(\delta) u) f (x + y + u \epsilon_i) dy \right| \, dx
\]
\[
\leq (\varphi_i(t))^{pl_i} \left| \int_{-\infty}^{+\infty} \zeta_i \left( \frac{u}{\varphi_i(t)}, \frac{\rho_i(\varphi_i(t), x)}{\varphi_i(t)} - \frac{1}{2} \rho_i'(\varphi_i(t), x) \right) \varphi_i(t)^{-l_i} \, du \right| \, dx
\]
\[
\times \Delta^m_{i} (\varphi_i(\delta) u, G_{\varphi_i(t)}) f du \left|_{p, G_{\varphi_i(t)}} \right| \left( \varphi_j(t) \right)^{\beta_j p}.
\]
Inequality (1.11) is proved analogously. □

Inequalities (1.12), (1.13) and (1.18) for \( r = q \) and for any \( \varpi \in U \) reduce to the estimation

\[
\|H^1_{i,j}\|_{qU \cap \ell(\varpi)} \leq C_1 \left\| (\varphi_i(t))^{-l_i} \Delta^m_i (\varphi_i(t)) f \right\|_{p,\varphi,\beta;G} \times \left| Q^j_{i} \prod_{j=1}^{n} (\psi_j(\xi))^{\beta_j} \right|, \quad (Q^j_{i} < \infty). \tag{1.19}
\]

In the case \( Q^j_{i,T} < \infty \) inequality (1.10) can be proved in the same way.

From inequality (1.18) for \( r = q \) and (1.19) we get the inequality \( (\forall \varpi \in U) \)

\[
\sup_{\varpi \in U} \|L_i\|_{qU \cap \ell(\varpi)} \leq C_2 \left\| (\varphi_i(t))^{-l_i} \Delta^m_i (\varphi_i(t), G_{\varphi(t)}) f \right\|_{p,\varphi,\beta;G} \prod_{j=1}^{n} (\psi_j(\xi))^{\beta_j},
\]

\[
\sup_{\varpi \in U} \|H^1_{i,j}\|_{qU \cap \ell(\varpi)} \leq C_3 \left\| (\varphi_i(t))^{-l_i} \Delta^m_i (\varphi_i(t), G_{\varphi(t)}) f \right\|_{p,\varphi,\beta;G} \prod_{j=1}^{n} (\psi_j(\xi))^{\beta_j}. \tag{1.20}
\]

From last inequalities it follows that

\[
\|L_i\|_{q,\psi,\beta;U} \leq C_4 \left\| (\varphi_i(t))^{-l_i} \Delta^m_i (\varphi_i(t), G_{\varphi(t)}) f \right\|_{p,\varphi,\beta;G}, \quad (2.1)
\]

\[
\|H^1_{i,j}\|_{q,\psi,\beta;U} \leq C_5 \left\| (\varphi_i(t))^{-l_i} \Delta^m_i (\varphi_i(t), G_{\varphi(t)}) f \right\|_{p,\varphi,\beta;G}. \tag{1.21}
\]

\( C_4 \) and \( C_5 \) are the constants independent of \( \varphi \).

### 2 Main Results

Prove two theorems on the properties of the functions from the space \( H^1_{p,\varphi,\beta}(G, \lambda) \).

**Theorem 2.1.** Let \( G \subset R^n \) satisfy the condition of flexible \( \varphi \)-horn, \( 1 \leq p \leq q \leq \infty, \nu = (\nu_1, \nu_2, ..., \nu_n), \nu_j \geq 0 \) be entire \( j = 1, 2, ..., n \), \( Q^j_{i,T} < \infty \) \( (i = 1, 2, ..., n) \) and let \( f \in H^1_{p,\varphi,\beta}(G, \lambda) \). Then the following embeddings hold

\[
D' : H^1_{p,\varphi,\beta}(G) \to L_{q,\psi,\beta;1}(G),
\]

more precisely, for \( f \in H^1_{p,\varphi,\beta}(G, \lambda) \) there exists a generalized derivative \( D' f \) and the following inequalities are valid

\[
\|D' f\|_{q,G} \leq C_1 (B(t)) \|f\|_{q,\psi,\beta;G} + \sum_{i=1}^{n} \left| Q^j_{i} \right| \sup_{0 < t < t_0} \left\| \Delta^m_i (\varphi_i(t), G_{\varphi(t)}) f \right\|_{p,\varphi,\beta;G}, \tag{2.1}
\]

\[
\|D' f\|_{q,\psi,\beta;G} \leq C_2 \|f\|_{H^1_{p,\varphi,\beta}(G, \lambda)}, \quad p \leq q < \infty. \tag{2.2}
\]

In particular, if

\[
Q^j_{T,0} = \int_{0}^{T} \prod_{j=1}^{n} (\varphi_j(t))^{-\nu_j - \frac{1}{2}(1-\beta_j) \frac{p}{2}} \frac{\varphi_j'(t)}{(\varphi_j(t))^{1/2}} dt < \infty, \quad (i = 1, 2, ..., n), \tag{2.3}
\]
then $D^\nu f(x)$ is continuous on $G$, i.e
\[
\sup_{x \in G} |D^\nu f(x)| \leq C_1 (B(t)\|f\|_{p,\varphi,\beta,G}) \\
+ \sum_{i=1}^{n} |Q_T^i| \sup_{0 < t < t_n} \|\Delta_{\nu}^{m_i} (\varphi_i(t), G\varphi_i(t) f)\|_{p,\varphi,\beta,G} 
\]

(2.4)

$0 < T \leq \min \{1, T_0\}$, $T_0$ is a fixed number; $C_1$, $C_2$, $C_3$, $C_4$ are the constants independent of $f$, also $C_1$ and $C_3$ are independent from $T$.

**Proof.** At first note that in the conditions of our theorem there exists a generalized derivative $D^\nu f$ on $G$. Indeed, from the condition $Q_{T_i} < \infty$ for all $(i = 1, 2, ..., n)$ it follows that for $f \in H_{p,\varphi,\beta}^1(G) \to H_{p,\varphi,\beta}^1(G)$, there exists $D^\nu f \in L_p(G)$ and for integral representation (1.3) and (1.4) with the same kernels is valid.

Applying the Minkowski inequality, from identities (1.3) and (1.4) we get
\[
\|D^\nu f\|_{q,G} \leq \left\|f^{(\nu)}(t)\right\|_{q,G} + \sum_{i=1}^{n} \|H_T^i\|_{q,G}. 
\]

(2.5)

By means of inequality (1.11) for $U = G$, $M_i = K_i^1$, $t = T$ we get
\[
\left\|f^{(\nu)}(t)\right\|_{q,G} \leq \|f\|_{p,\varphi,\beta,G} \prod_{j=1}^{n} (\varphi_j(t))^{-\nu_j - (1 - \beta_j)(\frac{1}{2} - \frac{1}{\beta_j})} \prod_{j=1}^{n} (\psi_j(\xi_1))^2 
\]

\[
\leq C_1 A(t)\|f\|_{p,\varphi,\beta,G}. 
\]

(2.6)

and by means of inequality (1.9) for $\eta = T$, $M_i = K_i^1$, $U = G$, we get
\[
\|H_T^i\|_{q,G} \leq C_2 Q_{T_i}^i \left\|\varphi_i(t)\right\|^{-l_i} \Delta_{\nu}^{m_i} (\varphi_i(t), G\varphi_i(t) f) \|_{p,\varphi,\beta,G}. 
\]

(2.7)

Substituting (2.7) and (2.6) in (2.5), we get inequality (1.21). By means of inequalities (1.20) and (1.21) for $\eta = T$ we get inequality (2.2).

Now let conditions (2.3) be satisfied, then take into account identities (1.3), (1.4), from inequality (2.5) we get
\[
\|D^\nu f - f^{(\nu)}(t)\|_{\infty,G} \leq C \sum_{i=1}^{n} |Q_T^i| \sup_{0 < t < t_n} \left\|\Delta_{\nu}^{m_i} (\varphi_i(t), G\varphi_i(t) f)\right\|_{p,\varphi,\beta,G}. 
\]

As $T \to 0$, the left side of this inequality tends to zero, since $f^{(\nu)}(t)(x)$ is continuous on $G$ and the convergence on $L_{\infty}(G)$ coincides with the uniform convergence. Then the limit function $D^\nu f$ is continuous on $G$.

Let $\gamma$ be an $n$-dimensional vector.

**Theorem 2.2.** Let all the conditions of Theorem 1 be fulfilled. Then for $Q_{T_i} < \infty$ $(i = 1, 2, ..., n)$ the derivative $D^\nu f$ satisfies on $G$ the Hölder generalized condition, i.e the following inequality is valid:
\[
\|\Delta (\gamma, G) D^\nu f\|_{q,G} \leq C\|f\|_{H_{p,\varphi,\beta}^1(G)} \cdot |h (|\gamma|, \varphi; T)|, 
\]

(2.8)

where $C$ is a constant independent of $f$, $|\gamma|$ and $T$.

In particular, if $Q_T < \infty$ $(i = 1, 2, ..., n)$, then
\[
\sup_{x \in G} |\Delta (\gamma, G) D^\nu f (x)| \leq C\|f\|_{H_{p,\varphi,\beta}^1(G)} \cdot |h_0 (|\gamma|, \varphi; T)|. 
\]

(2.9)

where $h (|\gamma|, \varphi, T) = \max_i \left\{ |\gamma|, Q_{|\gamma|}, Q_{|\gamma|, T} \right\}$ $h_0 (|\gamma|, \varphi, T) = \max_i \left\{ |\gamma|, Q_{|\gamma|,0}, Q_{|\gamma|, T,0} \right\}$

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Proof. According to Lemma 8.6 from [1] there exists a domain
\[ G_\omega \subset G \] (\omega = \xi r (x), \xi > 0 r (x) = \rho (x, \partial G), x \in G)
and assume that \(|\gamma| < \omega\), then for any \(x \in G_\omega\) the segment connecting the points \(x, x + \gamma\) is contained in \(G\). Consequently, for all the points of this segment, identities (1.3), (1.4) with the same kernels are valid. After the same transformations, from (1.3) and (1.4) we get
\[
|\Delta (\gamma, G) D^\nu f (x)| \leq \prod_{j=1}^n (\varphi_j(t))^{-1-\nu_j} \\
\times \int_R|f (x+y+z)| \left| \Omega^{(\nu)} \left( \frac{y-\gamma}{\varphi(t)}, \frac{\rho (\varphi(t), x)}{2 \varphi(t)} \right) \\
- \Omega^{(\nu)} \left( \frac{y}{\varphi(t)}, \frac{\rho (\varphi(t), x)}{2 \varphi(T)} \right) \right| dydz \\
+ C_0 \sum_{i=1}^n \left\{ \int_0^1 \int_{R^n} + \int_{R^n}^{-\infty} \left[ \zeta_i \left( \frac{u}{\varphi(t)}, \frac{\rho (\varphi(t), x)}{\varphi(t)} \right) - \frac{1}{2} \rho ' (\varphi(t), x) \right] \times \right. \\
\times \left| \frac{K_i^{(\nu)}}{K_i^{(\nu)}} \left( \frac{y}{\varphi(t)} \right) \right| \left| \Delta_m (\varphi_i (\delta) u f (x + y + u \zeta_i)) dydu dt \right. \\
+ \int_0^1 \left| \Delta_m (\varphi_i (\delta) u f (x + y + v \gamma)) d\nu du dy \right. \\
- C_1 A (x, \gamma) + C_2 \sum_{i=1}^n (E (x, \gamma) + F (x, \gamma),
\tag{2.10}
\end{align}
where \(0 < T \leq \{1, T_0\}\). Additionally, we assume that \(|\gamma| < T\). Consequently, \(|\gamma| < \min (\omega, T)\). If \(x \in G \setminus G_\omega\), then
\[ \Delta (\gamma, G) D^\nu f (x) = 0. \]

By inequality (2.9) we have
\[
\|\Delta (\gamma, G) D^\nu f\|_{q, G} \leq \|A (\cdot, \gamma)\|_{q, G_\omega} \\
+ \sum_{i=1}^n \left( \|E (\cdot, \gamma)\|_{q, G_\omega} + \|F (\cdot, \gamma)\|_{q, G_\omega} \right),
\tag{2.11}
\end{align}
\[
A (x, \gamma) \leq \prod_{j=1}^n (\varphi_j(t))^{-1-\nu_j} \int_0^1 \int_{R^n} |f (x + \zeta e_\gamma + y)| \\
\times \left| D_j \Omega^{(\nu)} \left( \frac{y}{\varphi(T)} \right) \frac{\rho (\varphi(t), x)}{2 \varphi(t)} \right| \zeta^{(\nu)} \left( \frac{z}{\varphi(T)} \right) \frac{\rho (\varphi(t), x)}{2 \varphi(t)} dx dz.
\]

Taking into account \(\xi e_\gamma + G_\omega \subset G\), and applying the generalized Minkowski inequality, from inequality (1.11) for \(U = G\), we have
\[
\|A (\cdot, \gamma)\|_{q, G_\omega} \leq C_1 \|\gamma\| \|f\|_{p, \varphi, \beta, G}. \tag{2.12}
\]

By means of inequality (1.9), for \(U = G\), \(\eta = |\gamma|\) we get
\[
\|E (\cdot, \gamma)\|_{q, G_\omega} \leq C_2 \|Q_i^{(\gamma)} \| (\varphi_i(t))^{-1} \Delta_m (\varphi_i(t), G_{\varphi(t)}) f \|_{p, \varphi, \beta, G}. \tag{2.13}
\]
and by means of inequality (1.10) for $U = G$, $\eta = |\gamma|$ we get
\[
\|F (\cdot, \gamma)\|_{q,G} \leq C_3 \left| Q_{[\gamma], T} \right| \left\| (\varphi_i(t))^{-1} \Delta^{\nu_i} (\varphi_i(t), G_{\varphi(t)}) f \right\|_{p,\varphi, \beta, G}.
\]  
(2.14)

From inequalities (2.11) and (2.12)-(2.14) we get the required inequality.

Now suppose that $|\gamma| \geq \min (\omega, T)$. Then
\[
\|\Delta (\gamma, G) D^\nu f\|_{q,G} \leq 2 \|D^\nu f\|_{q,G} \leq C (\omega T) \|D^\nu f\|_{q,G} |h (|\gamma|, \varphi; T)|.
\]

Estimating for $\|D^\nu f\|_{q,G}$ by means of inequality (2.1), in this case we get estimation (2.8). □

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References